Information Technology and Quantitative Management 2014

The majority preference relation based on cone preference relations of the decision makers

Alexey Zakharov

Saint-Petersburg State University
Faculty of Applied Mathematics and Control Processes,
Department of Control Theory,
Russia, Saint-Petersburg

4 июня 2014 г.

Individual multicriteria choice model of DM_1

X — set of feasible decisions, $X \subseteq \mathbb{R}^n$
$\mathbf{f} = (f_1, \ldots, f_m)$ — vector criterion, $\mathbf{f}$: $X \to \mathbb{R}^m$

Components of the individual model

- set of feasible vectors (outcomes) Y, $Y = \mathbf{f}(X)$
- individual binary preference relation $\succ_l$ of the DM_l.

Individual preference relation $\succ_l$

$\mathbf{y}^{(1)} \succ_l \mathbf{y}^{(2)} \iff \mathbf{y}^{(1)}$ is more preferred than $\mathbf{y}^{(2)}$

Individual preference relation $\succ_l$

Axioms of individual rational choice [Noghin, 2002]

1. The Pareto axiom.

$$\text{If } \mathbf{y}^{(1)} \geq \mathbf{y}^{(2)} \text{ than } \mathbf{y}^{(1)} \succ_l \mathbf{y}^{(2)}.$$

2. Axiom of invariance under a linear positive transformation.

$$\forall \mathbf{y}^{(1)},\ \mathbf{y}^{(2)} \in \mathbb{R}^m,\ \forall \mathbf{c} \in \mathbb{R}^m,\ \forall \alpha \in \mathbb{R},\ \alpha > 0$$
$$\mathbf{y}^{(1)} \succ_l \mathbf{y}^{(2)} \Rightarrow \alpha \mathbf{y}^{(1)} + \mathbf{c} \succ_l \alpha \mathbf{y}^{(2)} + \mathbf{c}.$$

Properties of relation $\succ_l$

$\succ_l$ — cone relation with cone K_l ($\mathbf{y}^{(1)} \succ_l \mathbf{y}^{(2)} \Leftrightarrow \mathbf{y}^{(1)} - \mathbf{y}^{(2)} \in K_l$)

Cone K_l:

1) convex
2) pointed
3) $\mathbb{R}^m_+ \subset K_l$, $\mathbf{0} \notin K_l$

Majority preference relation $\succ$

Consider n DMs and group preference relation $\succ$

Definition

$\mathbf{y}^{(1)} \succ \mathbf{y}^{(2)}$:
$\exists\{l_1, ..., l_p\} \subset \{1, ..., n\}: \quad \mathbf{y}^{(1)} \succ_{l_j} \mathbf{y}^{(2)} \quad \forall j \in \{1, ..., p\}$,
where $p = [(n+1)/2]$.

Lemma 1

Majority preference relation $\succ$ is cone relation with cone K, where

$$K = \bigcup_{l=1}^{C_n^p} \bigcap_{j=1}^{p} K_{lj},$$

$\{K_{l1}, ..., K_{lp}\} \subset \{K_1, ..., K_n\}$, $p = [(n+1)/2]$.

Properties of majority preference relation $\succ$

Lemma 2

Majority preference relation $\succ$:
1) irreflexive
2) invariant under a linear positive transformation
$\forall \mathbf{y}^{(1)},\ \mathbf{y}^{(2)} \in \mathbb{R}^m$, $\forall \mathbf{c} \in \mathbb{R}^m$, $\forall \alpha \in \mathbb{R}$, $\alpha > 0$
$\mathbf{y}^{(1)} \succ \mathbf{y}^{(2)} \Rightarrow \alpha \mathbf{y}^{(1)} + \mathbf{c} \succ \alpha \mathbf{y}^{(2)} + \mathbf{c}$.
3) cone K: $\mathbb{R}^m_+ \subset K$, $\mathbf{0} \notin K$

In general, cone K is not convex.
K — convex $\Leftrightarrow$ $\succ$ — transitive
Problem: Specify convex part of cone K
(transitive part of relation $\succ$)

Individual preferences

"Quantum" of information about relation $\succ_l$ [Noghin]

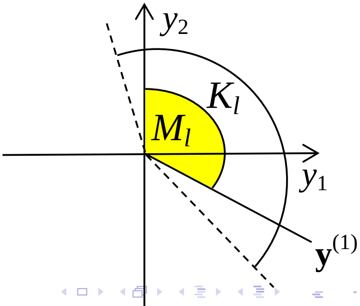
$$i \longrightarrow j \text{ with } w_i^{(l)}, w_j^{(l)} > 0$$
$$\mathbf{y} \in \mathbb{R}^m: \quad y_i = w_i^{(l)},\ y_j = -w_j^{(l)},\ y_s = 0\ \forall s \neq i, j$$
$$\mathbf{y} \succ_l \mathbf{0}$$

"Quantum" of information specify the known part (cone M_l) of cone K_l, $M_l \subseteq K_l$

DM$_l$ has "quanta" of information

$\downarrow$

cone M_l for each $l \in \{1, ..., n\}$,
$M_l \subseteq K_l$



Case of 3 DM

Let $n = 3$.

$\text{DM}_1 : \succ_1 \leftrightarrow \ K_1$, $\text{DM}_2 : \succ_2 \leftrightarrow \ K_2$, $\text{DM}_3 : \succ_3 \leftrightarrow \ K_3$.

Majority preference relation $\succ$ with cone K

$$K = (K_1 \cap K_2) \cup (K_1 \cap K_3) \cup (K_2 \cap K_3).$$

cone K: $\mathbb{R}^m_+ \subset K$, $0_m \notin K$

In general, cone K is not convex (relation $\succ$ is not transitive).

DM$_l$ for each $l \in \{1, 2, 3\}$ has "quanta" of information

$\downarrow$

cone M_l for each $l \in \{1, 2, 3\}$, $M_l \subseteq K_l$

$\downarrow$

cone $M = (M_1 \cap M_2) \cup (M_1 \cap M_3) \cup (M_2 \cap M_3)$ — known part of cone K

$\downarrow$

If M is convex, "optimal" choice — $Ndom_M(Y)$

If M is not convex, specify cone $\hat{M}$ — convex part of M, "optimal" choice — $Ndom_{\hat{M}}(Y)$

"Quantum" of information for each DM

Let $m = 2$, $\text{DM}_1 : \succ_1$, $\text{DM}_2 : \succ_2$, $\text{DM}_3 : \succ_3$.

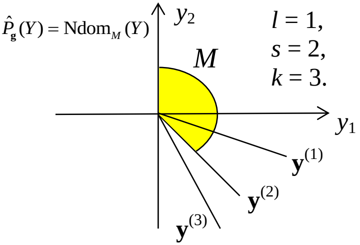
DM_1: $\mathbf{y}^{(1)} = \begin{pmatrix} w_1^{(1)} \\ -w_2^{(1)} \end{pmatrix} \succ_1 \mathbf{0}$, $M_1 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(1)}\} \setminus \mathbf{0}$.

DM_2: $\mathbf{y}^{(2)} = \begin{pmatrix} w_1^{(2)} \\ -w_2^{(2)} \end{pmatrix} \succ_2 \mathbf{0}$,

$M_2 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(2)}\} \setminus \mathbf{0}$.

DM_3: $\mathbf{y}^{(3)} = \begin{pmatrix} w_1^{(3)} \\ -w_2^{(3)} \end{pmatrix} \succ_3 \mathbf{0}$,

$M_3 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(3)}\} \setminus \mathbf{0}$.



Theorem 1

$\hat{P}_{\mathbf{g}}(Y) \subseteq P(Y)$, $\hat{P}_{\mathbf{g}}(Y) = \mathbf{f}(P_{\mathbf{g}}(X))$.
$\mathbf{g} : g_1 = f_1$, $g_2 = w_2^{(s)} f_1 + w_1^{(s)} f_2$.
$s : w_1^{(l)} w_2^{(s)} - w_2^{(l)} w_1^{(s)} > 0$, $w_1^{(s)} w_2^{(k)} - w_2^{(s)} w_1^{(k)} > 0$,
$l, s, k \in \{1, 2, 3\}$, $l \neq s$, $l \neq k$, $s \neq k$.

"Quantum" of information for each DM

Let $m = 2$, $\mathrm{DM}_1 : \succ_1$, $\mathrm{DM}_2 : \succ_2$, $\mathrm{DM}_3 : \succ_3$.

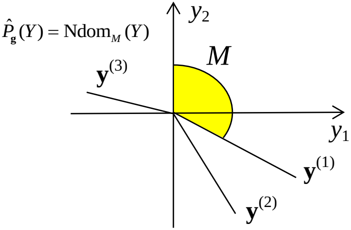
DM_1: $\mathbf{y}^{(1)} = \begin{pmatrix} w_1^{(1)} \\ -w_2^{(1)} \end{pmatrix} \succ_1 \mathbf{0}$, $M_1 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(1)}\} \setminus \mathbf{0}$.

DM_2: $\mathbf{y}^{(2)} = \begin{pmatrix} w_1^{(2)} \\ -w_2^{(2)} \end{pmatrix} \succ_2 \mathbf{0}$,

$M_2 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(2)}\} \setminus \mathbf{0}$.

DM_3: $\mathbf{y}^{(3)} = \begin{pmatrix} -w_1^{(3)} \\ w_2^{(3)} \end{pmatrix} \succ_3 \mathbf{0}$,

$M_3 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(3)}\} \setminus \mathbf{0}$.



Theorem 2

$\hat{P}_{\mathbf{g}}(Y) \subseteq P(Y)$, $\hat{P}_{\mathbf{g}}(Y) = f(P_{\mathbf{g}}(X))$.

$\mathbf{g} : g_1 = f_1$,

$g_2 = w_2^{(1)} f_1 + w_1^{(1)} f_2$, if $w_1^{(1)} w_2^{(2)} - w_2^{(1)} w_1^{(2)} > 0$,

$g_2 = w_2^{(2)} f_1 + w_1^{(2)} f_2$, if $w_1^{(1)} w_2^{(2)} - w_2^{(1)} w_1^{(2)} < 0$.

Two "quanta" of information for each DM

Let $m = 2$, $\mathrm{DM}_1 : \succ_1$, $\mathrm{DM}_2 : \succ_2$, $\mathrm{DM}_3 : \succ_3$.

DM_1: $\mathbf{y}^{(1)} = \begin{pmatrix} w_1^{(1)} \\ -w_2^{(1)} \end{pmatrix} \succ_1 \mathbf{0}$, $\bar{\mathbf{y}}^{(1)} = \begin{pmatrix} -v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} \succ_1 \mathbf{0}$,

$M_1 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(1)}, \bar{\mathbf{y}}^{(1)}\} \setminus \mathbf{0}$.

DM_2: $\mathbf{y}^{(2)} = \begin{pmatrix} w_1^{(2)} \\ -w_2^{(2)} \end{pmatrix} \succ_2 \mathbf{0}$, $\bar{\mathbf{y}}^{(2)} = \begin{pmatrix} -v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} \succ_2 \mathbf{0}$,

$M_2 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(2)}, \bar{\mathbf{y}}^{(2)}\} \setminus \mathbf{0}$.

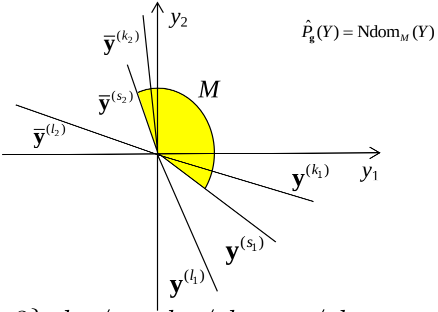
DM_3: $\mathbf{y}^{(3)} = \begin{pmatrix} w_1^{(3)} \\ -w_2^{(3)} \end{pmatrix} \succ_3 \mathbf{0}$, $\bar{\mathbf{y}}^{(3)} = \begin{pmatrix} -v_1^{(3)} \\ v_2^{(3)} \end{pmatrix} \succ_3 \mathbf{0}$,

$M_3 = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{y}^{(3)}, \bar{\mathbf{y}}^{(3)}\} \setminus \mathbf{0}$.

$$M = (M_1 \cap M_2) \cup (M_1 \cap M_3) \cup (M_2 \cap M_3)$$

cone $M \leftrightarrow$ known part of majority relation $\succ$

M is convex



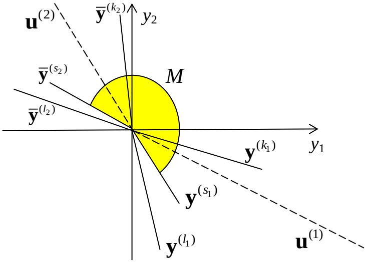
$l_1, s_1, k_1 \in \{1,2,3\}$, $l_1 \neq s_1$, $l_1 \neq k_1$, $s_1 \neq k_1$
$l_2, s_2, k_2 \in \{1,2,3\}$, $l_2 \neq s_2$, $l_2 \neq k_2$, $s_2 \neq k_2$
$w_1^{(s_1)} v_2^{(s_2)} - w_2^{(s_1)} v_1^{(s_2)} > 0$,

Theorem 3

$\hat{P}_{\mathbf{g}}(Y) \subseteq P(Y)$, $\hat{P}_{\mathbf{g}}(Y) = \mathbf{f}(P_{\mathbf{g}}(X))$.
$\mathbf{g} : g_1 = v_2^{(s_2)} f_1 + v_1^{(s_2)} f_2$,
$g_2 = w_2^{(s_1)} f_1 + w_1^{(s_1)} f_2$.

M is not convex

$l_1, s_1, k_1 \in \{1, 2, 3\}$,
$l_1 \neq s_1$, $l_1 \neq k_1$, $s_1 \neq k_1$
$l_2, s_2, k_2 \in \{1, 2, 3\}$,
$l_2 \neq s_2$, $l_2 \neq k_2$, $s_2 \neq k_2$



$\hat{M} = cone\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$,
$u_1^{(1)} u_2^{(2)} - u_2^{(1)} u_1^{(2)} > 0$,
$\hat{M}$ — convex part of cone M,
$\hat{M}$ is **not unique**.

Theorem 4

$\hat{P}_{\mathbf{g}}(Y) \subseteq P(Y)$, $\hat{P}_{\mathbf{g}}(Y) = \mathbf{f}(P_{\mathbf{g}}(X))$.
$\mathbf{g}: g_1 = u_2^{(2)} f_1 + u_1^{(2)} f_2$,
$g_2 = u_2^{(1)} f_1 + u_1^{(1)} f_2$.
$Ndom_{\hat{M}}(Y) = \hat{P}_{\mathbf{g}}(Y)$

Conclusions

1. Introduced the majority preference relation using cone preference relations of the DMs
2. Considered the properties of cone of the majority preference relation
3. Investigated case of 3 DM, aggregation of individual preferences ("quanta" of information) using the majority preference
4. Shown, that using such information the "optimal" group choice is the Pareto set of "new" multicriteria problem
5. In specific cases such "optimal" choice is not unique

References

1. Arrow, K. J. Social Choice and Individual Values. Wiley, New York, 1963.
2. Aizerman, M.A., Aleskerov, F.T. Vybor variantov: osnovy teorii (Variants choice: basis of the theory). Moscow: Nauka, 1990, in Russian.
3. Aleskerov, F.T., Habina, E.L., Shvarts, D.A. Binarnie otnoshenia, grafy i kollektivnye rechenia (Binary relations, graphs and collective decisions). Moscow: Izd. dom GU Vyschai Shkola Ekonomiki, 2006, in Russian.
4. Mirkin, B.G. Problema gruppovogo vybora (Problem of group choice). Moscow: Izdatel'stvo "Nauka", 1974, in Russian.
5. Petrovskii, A.B. Teoriya prinyatiya reshenii (Theory of Decision Making). Moscow: Akademia, 2009, in Russian.
6. Hwang Ch.-L., Lin M.-J. Group decision making under multiple criteria. Methods and applicatoins. Springer-Verlag, 1987.
7. Kilgour, D.M., Eden, C. Handbook of Group Decision and Negotiation. Springer, Dordrecht, 2010.
8. Ehrgott, M., Figueira, J.R., Greco S. Trends in Multiple Criteria Decision Analysis. Springer, 2010.

References

9. Noghin, V.D. Prinatie reshenii v mnogokriterial'noi srede: collichestvennyi podkhod (Decision Making in Multicriteria Sphere: Quantitative Approach), Moscow: Fizmatlit, 2005, in Russian.
10. Noghin, V.D. An algorithm of Pareto set reducing based on arbitrary finite collection of «quanta» of information. Iskusstvennyi intellekt i prinyatie reshenii, 2013, No. 1, pp. 63–69, in Russian.
11. Noghin V.D. Axiomatic approach to reduce the Pareto set: computational aspects. Moscow international conference on operational research (ORM2013), pp. 58–60.
12. Klimova, O.N., Noghin V.D. Using interdependent information on the relative importance of criteria in decision making. Computational Mathematics and Mathematical Physics, 2006, Vol. 46, No. 12, pp. 2080–2091.

Thank you very much!