Regime-dependent robust risk measures with application in portfolio selection

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Outline

- Introduction
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- Conclusions

Risk measure

Traditional risk measure

- The traditional risk measure can be regarded as a aggregation function $\rho : L_p(\mathscr{F}) \to R$ with respect to the probability P, here $1 \leq p < \infty$.
- The famous risk measure VaR can be described as follows:

$$VaR(x) = \min \gamma \quad \text{s.t.} \quad \text{Prob}\{\gamma \leq x\} \leq \epsilon,$$

  $\epsilon \in (0, 1]$ is a given loss tolerant probability (say, 5%).
- The computation of risk measure relies on the underlying distribution P of x.

Unknown distribution

- Traditional distribution assumptions, such as normal or student's t, does not fit the financial data well.
- Fully distributional information is hardly known in practice.

Deal with the unknown distribution

- Sample average approximation (Shapiro et al. [2009])
  Generate samples to represent the original distribution.
- Worst-case estimation (Bertsimas et al. [2011])
  Make decisions with the worst sample.
- Distributional robust (El Ghaoui et al. [2003])
  Finding a worst estimation among all possible known distributions.

Distributional robust risk measure

We can estimate ρ by assuming P belongs to an uncertainty set $\mathscr{P}$. This gives us the following worst-case risk measure:

Definition 1

For given risk measure ρ, the worst-case risk measure with respect to $\mathscr{P}$ is defined as $w\rho(x) \triangleq \sup_{P\in\mathscr{P}} \rho(x)$.

- By constructing different uncertainty sets $\mathscr{P}$, we can derive different versions of worst-case risk measures.
- Typical uncertainty sets proposed in the literature include the box uncertainty, the ellipsoidal uncertainty, and the mixture distribution uncertainty.

Researches on Distribution robust risk measure

Worst-case variance

- Lobo and Boyd [1999] proposed a worst-case analysis with respect to uncertain variance, and demonstrated the worst-case variance problem is a seme-definite program.

Worst-case VaR

- El Ghaoui et al. [2003] considered the worst-case value-at-risk (VaR) with uncertain first and second order moments, and showed the worst-case VaR constraint is equivalent to a second order cone constraint.

Researches on Distribution robust risk measure Mean-wrVaR model (Cont'd)

Worst-case CVaR

- Zhu and Fukushima [2009] considered the portfolio selection models with worst-case CVaR constraints, and solved it with discrete samples.
- Chen et al. [2011] considered the worst-case lower partial moments and worst-case conditional value-at-risk (CVaR) with respect to the first two order moments, and derived a tight bound for these two problems.

Regime-switching environment

Regime switching

- We consider the uncertainty set which is regime-dependent
- Regime switching describes the trend of macro economy and it can reflect dynamic correlations of return rates in different economic cycles
- We assume there are K regimes possibly appearing.
- We assume the regime switching is Markovian with the following transition probability matrix:

$$P^s = \begin{pmatrix} P^s_{s^1 s^1} & P^s_{s^1 s^2} & \cdots & P^s_{s^1 s^K} \\ P^s_{s^2 s^1} & P^s_{s^2 s^2} & \cdots & P^s_{s^2 s^K} \\ \cdots & \cdots & \cdots & \cdots \\ P^s_{s^K s^1} & P^s_{s^K s^2} & \cdots & P^s_{s^K s^K} \end{pmatrix},$$

Regime-dependent risk measure

- We assume that uncertainty sets $\mathscr{P}(s)$ are associated with the possible regime $s \in S$, S is the regime set.
- Given a particular regime s, the regime-dependent worst-case risk measure can be defined as

$$w\rho^s(x) \triangleq \sup_{P \in \mathscr{P}(s)} \rho(x).$$

Combine the sub risks into one

- One takes the greatest risk measure value among all the possible regimes
- The other mixes the sub risks together with respect to their occurring probabilities

Regime-dependent risk measure (Cont'd)

We define them as follows:

Worst regime risk measure

For the risk measure ρ, the worst regime risk measure with respect to the regime set S is defined as

$$w r\rho(x) \triangleq \sup_{s\in S} w\rho^{s}(x) = \sup_{s\in S} \sup_{P\in\mathscr{P}(s)} \rho(x).$$

Mixed worst-case risk measure

For the risk measure ρ, the mixed worst-case risk measure with respect to the regime set S is defined as

$$m w\rho(x) \triangleq E_s[w\rho^{s}(x)] = E_s[\sup_{P\in\mathscr{P}(s)} \rho(x)] = \sum_s P^{s}_{s_0,s}(\sup_{P\in\mathscr{P}(s)} \rho(x).\ )$$

Properties of the regime-dependent risk measures

Proposition 1

Worst regime risk measure is equivalent to worst-case risk measure with respect to the uncertainty set $\bar{\mathscr{P}} = \bigcup_{s \in S} \mathscr{P}(s)$

$$wr\rho(x) = \sup_{P \in \bar{\mathscr{P}}} \rho(x).$$

Theorem of coherency

If $\rho(\cdot)$ is a coherent risk measure, then $wr\rho(\cdot)$ and $mw\rho(\cdot)$ associated with any regime set S and uncertainty sets $\mathscr{P}(s)$, $s \in S$, are both coherent risk measures.

Application to portfolio selection problem

Market setting

- There are n risky assets in the security market
- $w = [w_1, w_2, ..., w_n]^{\mathrm{T}}$: The proportion vector of the wealth invested in n assets
- $r = [r_1, r_2, ..., r_n]^{\mathrm{T}}$: The random return rates of n assets
- $x = -r^{\mathrm{T}}w$: The loss function
- Regime-dependent uncertainty set:

$$\mathscr{P}(s) = \{P | E_P[\xi] = \mu(s), \sigma_P^2(\xi) = \Gamma(s)\}, \ s \in S, \tag{1}$$

Such uncertainty set contains all possible distribution with given mean and variance under particular regime.

Mean-wrVaR model

The objective function:

$$\max_{w} E(x) - \lambda \cdot wrVaR(x) = \max_{w} E(x) - \lambda \cdot \sup_{s \in S} \sup_{P \in \mathscr{P}(s)} VaR_P(x). \quad (2)$$

Constraints on portfolio:

$$e^T w = 1, \quad (3)$$

$$\underline{w} \le w_i \le \overline{w}, \ i = 1, ..., n, \quad (4)$$

Mean-wrVaR model (Cont'd)

By introducing an auxiliary variable y, Mean-wrVaR model (2-4) can be expressed as:

$$\max \quad E(-r^{\mathrm{T}}w) - \lambda y \tag{5}$$

$$\text{s.t.} \quad \sup_{P \in \mathscr{P}(s)} VaR_P(-r^{\mathrm{T}}w) \leq y, \;\; s \in S, \tag{6}$$

$$e^T w = 1, \tag{7}$$

$$\underline{w} \leq w_i \leq \overline{w}, \;\; i = 1, \ldots, n. \tag{8}$$

Mean-wrVaR model (Cont'd)

By transforming the worst-case VaR constraint (6) with respect to the uncertainty set (1), (5-8) is equivalent to

$$
(SOCP1): \quad \begin{array}{ll} \max & E_s(\mu(s))^{\mathrm{T}} w - \lambda y \\ \text{s.t.} & \kappa(\epsilon) \|\Gamma^{1/2}(s) w\|_2 - \mu^{\mathrm{T}}(s) w \leq y, \ s \in S, \\ & e^T w = 1, \\ & \underline{w} \leq w_i \leq \overline{w}, \ i = 1, ..., n, \end{array}
$$

which is a second order cone programming.

Mean-mwVaR model

Replacing wrVaR in (2) by mwVaR, we have the mean-wrVaR portfolio selection model

$$\max \quad E(-r^{\mathrm{T}}w) - \lambda \cdot E_s[\sup_{P\in\mathscr{P}(s)} VaR_P(-r^{\mathrm{T}}w)], \tag{9}$$

$$\text{s.t.} \quad e^T w = 1, \tag{10}$$

$$\underline{w} \leq w_i \leq \overline{w}, \; i = 1, ..., n. \tag{11}$$

Mean-mwVaR model (Cont'd)

Again, problem (9)-(11) can be transformed into the following second order cone program:

$$(SOCP2): \quad \begin{array}{ll} \max & E_s(\mu(s) - y(s)) \\ \text{s.t.} & \kappa(\epsilon)\|\Gamma^{1/2}(s)w\|_2 - \mu^{\mathrm{T}}(s)w \leq y(s), \ s \in S, \\ & e^T w = 1, \\ & \underline{w} \leq w_i \leq \overline{w}, \ i = 1, \ldots, n. \end{array}$$

SOCP can be efficiently solved by some commercial optimization softwares, such as MOSEK.

Empirical illustrations

- We choose 10 stocks from different industries in American stock markets
- We use adjusted daily close-prices of these stocks on every Monday to compute their weekly logarithmic return rates from February 14, 1977 to January 30, 2012
- We divide the market into three regimes: the bull regime; the consolidation regime and the bear regime
- Assume the regime transition probability is stationary:

$$P = \begin{bmatrix} 0.9475 & 0.0336 & 0.0189 \\ 0.3333 & 0.3148 & 0.3519 \\ 0.0471 & 0.0634 & 0.8895 \end{bmatrix}.$$

Statistics information under three regimes

Table: expected return rates (%) under different regimes and the total sample

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
$\mu(s^1)$	0.099	0.068	0.112	0.095	0.078	0.101	0.116	0.076	0.115	0.110
$\mu(s^2)$	0.026	-0.079	0.134	-0.042	-0.142	0.108	0.172	0.070	0.170	0.1235
$\mu(s^3)$	-0.205	-0.185	0.094	-0.161	0.010	-0.076	-0.037	-0.261	-0.029	0.031
μ	0.099	0.068	0.112	0.095	0.078	0.101	0.116	0.076	0.115	0.1105

Both first and second order moments have significant difference among different regimes. (The estimated covariance matrices are omitted.)

Optimal portfolios

- Then we can find the optimal portfolios of mean-wrVaR, mean-mwVaR models by solving (SOCP1), (SOCP2).
- Besides, we show the optimal portfolio of mean-wVaR model in el Ghaoui et al. [2003] as a comparison.

Table: optimal portfolios under wrVaR, mvVaR and wVaR

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
$w^*_{wVaR}(s_0)$	0.000	0.000	0.300	0.000	0.300	0.000	0.000	0.000	0.260	0.140
$w^*_{wrVaR}(s_0)$	0.000	0.000	0.300	0.000	0.300	0.000	0.000	0.000	0.154	0.245
$w^*_{mwVaR}(s^1)$	0.000	0.000	0.300	0.000	0.300	0.000	0.087	0.000	0.300	0.012
$w^*_{mwVaR}(s^2)$	0.000	0.000	0.300	0.000	0.300	0.000	0.000	0.000	0.300	0.100
$w^*_{mwVaR}(s^3)$	0.000	0.000	0.300	0.000	0.300	0.000	0.000	0.000	0.246	0.153

The confidence level is $\epsilon = 0.05$, the trade-off parameter is $\lambda = 1$, the lower bound and upper bound of portfolio weights are 0 and 0.3, respectively.

Optimal portfolios (Con'd)

- Both the optimal portfolios of mean-wVaR model and mean-wrVaR model do not rely on the current regime.
- The mean-mwVaR model provides us with three optimal portfolios under three different regimes.
- That is because the estimation of mwVaR relies on the regime appearing probability in the future.
- The strategy derived under such mixed robust models reveals more information about market regimes than the traditional worst-case risk measures.

Performance of the optimal portfolios

- We compare the performance of three robust models by computing the expected return rate, the variance of the optimal portfolio, and the optimal estimation of the robust VaR.

Table: robust VaR value, expected return and variance under different robust portfolio selection models

	wVaR(s_0)	wrVaR(s_0)	mwVaR(s^1)	mwVaR(s^2)	mwVaR(s^3)
robust VaR value	0.0891	0.1078	0.0809	0.0923	0.1053
expected return (%)	0.1028	0.1023	0.1035	0.1030	0.1027
variance	6.3332e-5	6.2984e-5	6.4675e-5	6.4770e-5	6.3017e-5

Performance of the optimal portfolios (Con'd)

- The largest estimation on the robust VaR is attained under the mean-wrVaR model. And it leads to the most conservative strategy.
- The estimation of robust VaR has significant difference with respect to three regimes under the mean-mwVaR model.
- The expected return of mean-mwVaR model under the bull or consolidation regime is significantly larger than that of either the mean-wVaR model or the mean-wrVaR model, while the corresponding variance does not increase too much with regard to its magnitude
- wrVaR is suitable for conservative investors; mwVaR is more suitable for investors who focus on the market trend.

Conclusions

- We propose in this paper two classes of robust risk measures, which are time-dependent and coherent.
- We apply VaR to construct robust portfolio selection models and show that they can be transformed into second order cone programs.
- Empirical illustrations show that our new models can flexibly reflect the influence of different market regimes on the investment return and risk.
- We can also adopt other risk measures, such as CVaR, as the basic risk measure $\rho(\cdot)$ in our new risk measure definitions to construct other robust risk measures.
- Another interesting topic is to extend the results in this paper to the multi-period situation.

Thank You Very Much for Your Attention!