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NOTES ON DEPENDENT ATTRIBUTES IN TOPSIS

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INTRODUCTION

- **TOPSIS** (**T**echnique for **O**rder **P**reference by **S**imilarity to **I**deal **S**olution) considers the distances to the ideal and anti-ideal solutions using the **Euclidean norm** (Hwang and Yoon, 1981).
- **How** do we deal with the dependency of the attributes?
- **Modification (TOPSIS-M)**
  - A **new measurement** of ideal and anti-ideal distances based on the **Mahalanobis** distance
  - A **new method for synthesizing** the contribution of the two distances in the final ordering that allows the consideration of both aspects without the problems associated with a quotient

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INTRODUCTION

1. Introduction
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Multicriteria Decision Making Techniques

- **Definition**: a series of models, methods and techniques that allow a more effective and realistic solution to complex problems that contemplate multiple scenarios, actors and criteria (tangible and intangible).

- **Classification**:

  a) The flow of information:
  - (i) Techniques **without *a priori* information** on the preferences
  - (ii) Techniques **with *a priori* information**
  - (iii) Interactive techniques

  b) Set of alternatives:
  - (i) Continuous: **Multi-objective** Programming
  - (ii) Discrete: **Multi-attribute** Programming

  c) The approach
  - **(i) Efficient** solutions
  - (ii) Minimisation of **distances** (CP, GP, TOPSIS etc.)
  - **(iii) Value** function (MAUT, AHP etc.)
  - **(iv) Outranking** methods (ELECTRE, PROMETHEE etc.)

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Techniques based on distance minimisation .

- **Compromise Programming:**

$$Min_{x \epsilon X}\, d(z(x), z^*, p) = Min_{x \epsilon X} \left(\sum_{j=1}^{q} w_j^p \left| z_j^* - z_j(x) \right|^p \right)^{1/p}$$

$$z_j^*(x) = Max_{x \epsilon X}\, z_j(x)$$ **p=1, 2,…**

$$Min_{x \epsilon X}\, d(z(x), z^*, p = \infty) = Min_{x \epsilon X} Max_j \left\{ w_j \left| z_j^* - z_j(x) \right| \right\},$$
$$j = 1, \dots, q$$ **p=∞**

- **Goal programming** (Satisficing)
- **Vikor, TOPSIS…**
- **TOPSIS**
  - Traditional approach (Hwang & Yoon, 1981)
  - TOPSIS-M

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TRADITIONAL TOPSIS APPROACH

➢ Decision table:

- A_i (i=1,…,m), C_j (j=1,…,n) and w_j ($w_j > 0$, $\Sigma_j w_j = 1$)

	w1	w2	…	wj		wn
	C1	**C2**	**…**	**Cj**	**…**	**Cn**
A1	x_{11}	x_{12}	…	x_{1j}	…	x_{1n}
…	…	…	…	…	…	…
Ai	x_{i1}	x_{i2}	…	x_{ij}	…	x_{in}
…	…	…	…	…	…	…
Am	x_{m1}	x_{m2}	…	x_{mj}	…	x_{mn}

➢ Step 1. Calculate the **normalized decision matrix**

$$n_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^{2}}}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

➢ Step 2. Calculate the **weighted normalized** decision matrix

$$v_{ij} = w_j \cdot n_{ij}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

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TRADITIONAL TOPSIS APPROACH

- **Step 3**. Determine the "**positive ideal**" and "**negative ideal**" alternatives

$$A^+ = \{v_1^+, \ldots, v_n^+\},\ \text{where } v_i^+ = \max_i v_{ij}, i = 1, \ldots, m, j = 1, \ldots, n,$$

$$A^- = \{v_1^-, \ldots, v_n^-\},\ \text{where } v_i^- = \min_i v_{ij}, i = 1, \ldots, m, j = 1, \ldots, n.$$

- **Step 4**. Calculate the **distances**

$$d_i^+ = \left(\sum_{j=1}^{n} \left|v_j^+ - v_{ij}\right|^2\right)^{1/2},\ i = 1, \ldots, m$$

$$d_i^- = \left(\sum_{j=1}^{n} \left|v_j^- - v_{ij}\right|^2\right)^{1/2},\ i = 1, \ldots, m$$

- **Step 5**. Calculate the **relative proximity** to the ideal solution: $R_i = \frac{d_i^+}{d_i^+ + d_i^-},\ i = 1, \ldots, m$ $(0 \le R_i \le 1)$.

- **Step 6**. **Preference order** $(A_i \succ A_j \Leftrightarrow R_i < R_j)$

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TOPSIS-M (Mahalanobis) APPROACH

➢ Mahalanobis Distance

$$d_M(x,y) = \left((x-y)^T \Sigma^{-1}(x-y)\right)^{1/2}$$

$$\Sigma = \frac{1}{n-1}(X_c)^T(X_c)$$

$$X_c = (X - \bar{x})$$

X (mxn)

➢ TOPSIS and TOPSIS-M (independent attributes)

Table 2. Results obtained by Principal Component Analysis

Alternatives	F1	F2	F3	F4	F5
A_1	1,0151	-0,8463	0,001	1,4672	0,5175
A_2	-0,8201	1,5214	-0,0746	1,0127	-0,3852
A_3	-0,2968	0,0543	-1,385	-0,7713	1,25
A_4	-0,5684	-1,0871	-0,5394	-0,2665	-1,5165
A_5	1,4887	0,7791	0,3708	-0,962	-0,5298
A_6	-0,8185	-0,4214	1,6272	-0,4801	0,664

Table 3. R_i for traditional TOPSIS and TOPSIS-M

	TOPSIS		TOPSIS-M	
	R_i	Rank	R_i	Rank
A_1	0,4335	1	0,4335	1
A_2	0,4738	2	0,4738	2
A_3	0,5915	5	0,5915	5
A_4	0,8206	6	0,8206	6
A_5	0,4826	3	0,4826	3
A_6	0,4885	4	0,4885	4

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TOPSIS (dependent attributes)

➢ TOPSIS (Manhattan distance – p=1)

$$\varphi = \frac{\sum_{i=1}^{n}\sum_{j=i}^{n} r_{ij}^{2} - n}{n(n-1)}$$

Alternatives	C1	C2	C3	C4	C5
A_1	690	3.1	9	7	4
A_2	590	3.9	7	6	10
A_3	600	3.6	8	8	7
A_4	620	3.8	7	10	6
A_5	700	2.8	10	4	6
A_6	650	4	6	9	8

Gleason-Staelin redundancy measure (Phi) is ϕ= 0.6736 > 0.5 (threshold for dependency)

The resulting rankings for the different normalization modes are:
A6≻A2≻A4≻A3≻A1≻A5 for DM and EM
A6≻A4≻A2≻A3≻A1≻A5 for IM, UM and SS
A5≻A1≻A6≻A4≻A3≻A2 for non-normalized data

Table 5. R_i for the Manhattan distance with various normalization modes (DM, EM, IM, UM, EE, NN)

Alternatives	DM	Rank	EM	Rank	IM	Rank	UM	Rank	SS	Rank	NN	Rank
A1	0,6200	5	0,6178	5	0,6024	5	0,5182	5	0,5332	5	0,1643	2
A2	0,4151	2	0,4166	2	0,4288	3	0,5000	3	0,4866	3	0,9206	6
A3	0,4548	4	0,4555	4	0,4583	4	0,5152	4	0,5071	4	0,8443	5
A4	0,4230	3	0,4239	3	0,4201	2	0,4621	2	0,4616	2	0,6855	4
A5	0,6414	6	0,6388	6	0,6319	6	0,5333	6	0,5435	6	0,0881	1
A6	0,3744	1	0,3757	1	0,3750	1	0,3909	1	0,3938	1	0,4481	3

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TOPSIS (dependent attributes)

➢ TOPSIS (Euclidean distance – p=2)

Alternatives	C1	C2	C3	C4	C5
A_1	690	3.1	9	7	4
A_2	590	3.9	7	6	10
A_3	600	3.6	8	8	7
A_4	620	3.8	7	10	6
A_5	700	2.8	10	4	6
A_6	650	4	6	9	8

The resulting rankings for the different normalization modes ar
A6≻A2≻A4≻A3≻A1≻A5 for DM and EM
A6≻A4≻A2≻A3≻A1≻A5 for IM
A6≻A4≻A2≻A1≻A3≻A5 for UM
A6≻A4≻A2≻A3≻A5≻A1 for SS
A5≻A1≻A6≻A4≻A3≻A2 for non-normalized data

Table 6. R_i for the Euclidean distance (ED) with various normalizations (DM, EM, IM, UM, EE, NN)

	DM	Rank	EM	Rank	IM	Rank	UM	Rank	SS	Rank	NN	Rank
A1	0.6321	5	**0.6297**	**5**	0.6116	5	0.5127	4	0.5335	6	0.1080	2
A2	0.4130	2	**0.4144**	**2**	0.4295	3	0.5000	3	0.4832	3	0.9443	6
A3	0.4382	4	**0.4388**	**4**	0.4379	4	0.5129	5	0.4998	4	0.8979	5
A4	0.4355	3	**0.4364**	**3**	0.4280	2	0.4729	2	0.4727	2	0.7231	4
A5	0.6340	6	**0.6307**	**6**	0.6223	6	0.5183	6	0.5299	5	0.0623	1
A6	0.3869	1	**0.3896**	**1**	0.3944	1	0.4263	1	0.4298	1	0.4541	3

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TOPSIS (dependent attributes)

➢ TOPSIS (Tchebicheff distance – p=∞)

Alternatives	C1	C2	C3	C4	C5
A_1	690	3.1	9	7	4
A_2	590	3.9	7	6	10
A_3	600	3.6	8	8	7
A_4	620	3.8	7	10	6
A_5	700	2.8	10	4	6
A_6	650	4	6	9	8

The resulting rankings for the different normalization modes are
A2 ≻ A4 ≻ A6 ≻ A3 ≻ A5 ≻ A1 for DM and EM
A2 = A4 ≻ A3 ≻ A6 ≻ A5 ≻ A1 for IM
A4 ≻ A2 = A5 = A6 ≻ A1 ≻ A3 for UM
A4 ≻ A2 ≻ A5 ≻ A6 ≻ A3 ≻ A1 for SS
A5 ≻ A1 ≻ A6 ≻ A4 ≻ A3 ≻ A2 for non-normalized data

Table 7. R_i for the Tchebycheff distance (TD) with various normalizations (DM, EM, IM, UM, EE, NN)

	DM	Rank	EM	Rank	IM	Rank	UM	Rank	SS	Rank	NN	Rank
A1	0.6822	6	0.6820	6	0.6667	6	0.5238	5	0.5760	6	0.0909	2
A2	0.3832	1	0.3834	1	0.4000	1	0.5000	2	0.4474	2	0.9483	6
A3	0.4459	4	0.4457	4	0.4286	3	0.5769	6	0.5388	5	0.9091	5
A4	0.4171	2	0.4168	2	0.4000	1	0.4286	1	0.4232	1	0.7273	4
A5	0.6157	5	0.6109	5	0.6000	5	0.5000	2	0.5055	3	0.0517	1
A6	0.4282	3	0.4332	3	0.4444	4	0.5000	2	0.5210	4	0.4545	3

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TOPSIS (dependent attributes)

➢ TOPSIS (Mahalanobis distance – p=∞)

Alternatives	C1	C2	C3	C4	C5
A_1	690	3.1	9	7	4
A_2	590	3.9	7	6	10
A_3	600	3.6	8	8	7
A_4	620	3.8	7	10	6
A_5	700	2.8	10	4	6
A_6	650	4	6	9	8

The resulting ranking for the different normalization modes is:
A3≻A6≻A1≻A2≻A5≻A4
for any normalization type and non-standardized data

Table 8. R_i for the Mahalanobis distance (ED) with various normalizations (DM, EM, IM, UM, EE, NN)

	DN	Or	EN	Or	IM	Or	UM	Or	EE	Or	NN	Or
A1	0.4713	3	0.4713	3	0.4713	3	0.4713	3	0.4713	3	0.4713	3
A2	0.4726	4	0.4726	4	0.4726	4	0.4726	4	0.4726	4	0.4726	4
A3	0.4699	1	0.4699	1	0.4699	1	0.4699	1	0.4699	1	0.4699	1
A4	0.4743	6	0.4743	6	0.4743	6	0.4743	6	0.4743	6	0.4743	6
A5	0.4727	5	0.4727	5	0.4727	5	0.4727	5	0.4727	5	0.4727	5
A6	0.4709	2	0.4709	2	0.4709	2	0.4709	2	0.4709	2	0.4709	2

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RESULTS &CONCLUSIONS

- If the **attributes** are **independent**, as previously mentioned, the **values** obtained with traditional **TOPSIS** and **TOPSIS-M** are exactly the same.
- The **rankings** obtained when using traditional **TOPSIS** (distance and Euclidean normalization) and **TOPSIS-M** (Mahalanobis distance and non-normalized data) are **clearly different** when there is certain **dependence** with regards to the data, **even** in the case of attributes **with close to null dependence** (Gleason-Staelin's $\phi < 0.0230$)

Rank for ED						Rank for MD						
A1	A2	A3	A4	A5	A6	A1	A2	A3	A4	A5	A6	Phi
1	**3**	5	6	**2**	**4**	1	**2**	5	6	**4**	**3**	**0.022561**
1	3	5	6	4	2	1	2	5	6	4	3	0.050330
2	1	5	6	4	**3**	**3**	1	5	6	4	**2**	**0.220387**
2	1	5	6	4	3	3	1	5	6	4	2	0.221781
2	1	5	6	4	3	3	1	4	6	5	2	0.274012
2	3	5	6	1	4	3	1	5	6	2	4	0.282283

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RESULTS &CONCLUSIONS

- This result is **valid for** any of the Minkowski's distances **(p=1,2,…∞).**
- The data **normalization mode** that is followed for the **Minkowski distances conditions** the results obtained. This **does not occur** with the **Mahalanobis distance** as the results are the same if the initial data are normalized or not, irrespective of the type of normalization that is employed.
- If **the Manhattan distance** ($p = 1$) is used, the denominator of expression

$$R_i = \frac{d_i^+}{d_i^+ + d_i^-}, \quad i = 1, \dots, m$$

is constant ($d_i^+ + d_i^- = K, i = 1, \dots, m$), so the ranking given by the measurement of relative proximity (R_i) is the same as that given by the ideal distance ($Min_i\, d_i^+$) and the anti-ideal distance ($Max_i\, d_i^-$).

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RESULTS &CONCLUSIONS

- This result seems to be true for the considered example also for the **Mahalanobis** distance (**$\mathrm{Min}_i\ R_i \Leftrightarrow \mathrm{Min}_i\ d_i^+$**)
- As is well known, that the **Minkowski distances diminish** as the ***p*** **order** of the augmented norm **increases** $(\|\cdot\|_1 \geq \|\cdot\|_2 \geq \cdots \geq \|\cdot\|_\infty)$.
- With a **fixed norm order $p = 1, 2, \ldots, \infty$**, the **Minkowski distances** to the **ideal** and **anti-ideal increase** with the different **normalization modes**, in the following manner:
$$\|\cdot\|_p^{DM} \leq \|\cdot\|_p^{EM} \leq \|\cdot\|_p^{IM} \leq\leq \|\cdot\|_p^{NN}$$
- This latter **result** is **not verified** by relative proximity **(R_i)** for the problem that presents the synthesis of the ideal and anti-ideal distances as a quotient.

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Evolution of distances in accordance with normalization mode

$d(A^+)$

	p=1								p=2								p=∞								MH							
	Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without	
A1	0,0457	5	0,1089	5	0,2043	5	4,7300	2	0,0260	5	0,0617	5	0,1121	5	3,1757	2	0,0220	6	0,0519	6	0,0900	5	3,0000	2	331,5376	3	331,5376	3	331,5376	3	331,5376	3
A2	0,0359	3	0,0863	3	0,1721	3	34,2200	6	0,0206	4	0,0493	4	0,0972	4	33,0109	6	0,0136	2	0,0323	2	0,0600	2	33,0000	6	332,4918	4	332,4918	4	332,4918	4	332,4918	4
A3	0,0379	4	0,0908	4	0,1779	4	31,2300	5	0,0177	1	0,0424	1	0,0822	1	30,0076	5	0,0110	1	0,0259	1	0,0450	1	30,0000	5	330,5831	1	330,5831	1	330,5831	1	330,5831	1
A4	0,0355	2	0,0852	2	0,1643	2	25,2400	4	0,0205	3	0,0490	3	0,0921	3	24,0150	4	0,0146	3	0,0346	3	0,0600	2	24,0000	4	333,6375	6	333,6375	6	333,6375	6	333,6375	6
A5	0,0464	6	0,1105	6	0,2100	6	1,7400	1	0,0276	6	0,0655	6	0,1237	6	1,1080	1	0,0205	5	0,0484	5	0,0900	5	0,9000	1	332,5098	5	332,5098	5	332,5098	5	332,5098	5
A6	0,0316	1	0,0760	1	0,1464	1	16,2500	3	0,0192	2	0,0463	2	0,0894	2	15,0251	3	0,0170	4	0,0411	4	0,0800	4	15,0000	3	331,2310	2	331,2310	2	331,2310	2	331,2310	2

$d(A^-)$

	p=1								p=2								p=∞								MH							
	Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without	
A1	0,0336	5	0,0809	5	0,1629	5	31,1100	2	0,0183	6	0,0441	6	0,0877	6	30,0094	2	0,0128	6	0,0308	6	0,0600	5	30,0000	2	371,9658	3	371,9658	3	371,9658	3	371,9658	3
A2	0,0434	3	0,1035	3	0,1950	3	1,6200	6	0,0256	2	0,0608	2	0,1115	2	0,9942	6	0,0220	1	0,0519	1	0,0900	1	0,9000	6	371,0032	4	371,0032	4	371,0032	4	371,0032	4
A3	0,0414	4	0,0990	4	0,1893	4	4,6100	5	0,0209	4	0,0498	4	0,0940	5	3,1222	5	0,0136	5	0,0323	5	0,0600	5	3,0000	5	372,8841	1	372,8841	1	372,8841	1	372,8841	1
A4	0,0438	2	0,1046	2	0,2029	2	10,6000	4	0,0242	3	0,0575	3	0,1098	3	9,0543	4	0,0205	2	0,0484	2	0,0900	1	9,0000	4	369,8337	6	369,8337	6	369,8337	6	369,8337	6
A5	0,0329	6	0,0793	6	0,1571	6	34,1000	1	0,0204	5	0,0493	5	0,0976	4	33,0111	1	0,0170	4	0,0411	3	0,0800	3	33,0000	1	370,9547	5	370,9547	5	370,9547	5	370,9547	5
A6	0,0477	1	0,1139	1	0,2207	1	19,5900	3	0,0256	1	0,0609	1	0,1161	1	18,0272	3	0,0170	3	0,0403	4	0,0750	4	18,0000	3	372,2431	2	372,2431	2	372,2431	2	372,2431	2

(RC) Ranking

	p=1								p=2								p=∞								MH							
	Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without		Distrib.		Euclid.		Ideal		Without	
A1	0,5762	5	0,5736	5	0,5564	5	0,1320	2	0,5866	6	0,5833	6	0,5612	6	0,0957	2	0,6323	6	0,6273	6	0,6000	6	0,0909	2	0,4713	3	0,4713	3	0,4713	3	0,4713	3
A2	0,4528	3	0,4547	3	0,4689	3	0,9548	6	0,4458	2	0,4481	2	0,4658	3	0,9708	6	0,3832	1	0,3834	1	0,4000	1	0,9735	6	0,4726	4	0,4726	4	0,4726	4	0,4726	4
A3	0,4774	4	0,4785	4	0,4844	4	0,8714	5	0,4591	4	0,4604	4	0,4665	4	0,9058	5	0,4459	3	0,4457	3	0,4286	3	0,9091	5	0,4699	1	0,4699	1	0,4699	1	0,4699	1
A4	0,4478	2	0,4490	2	0,4475	2	0,7042	4	0,4586	3	0,4601	3	0,4560	2	0,7262	4	0,4171	2	0,4168	2	0,4000	1	0,7273	4	0,4743	6	0,4743	6	0,4743	6	0,4743	6
A5	0,5851	6	0,5821	6	0,5720	6	0,0485	1	0,5747	5	0,5709	5	0,5590	5	0,0325	1	0,5458	5	0,5407	5	0,5294	5	0,0265	1	0,4727	5	0,4727	5	0,4727	5	0,4727	5
A6	0,3989	1	0,4002	1	0,3988	1	0,4534	3	0,4292	1	0,4319	1	0,4348	1	0,4546	3	0,4996	4	0,5047	4	0,5161	4	0,4545	3	0,4709	2	0,4709	2	0,4709	2	0,4709	2

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RESULTS &CONCLUSIONS

- Using AHP to eliminate the problems associated with the ratio and to capture the relative importance of both distances (ideal and ante-ideal).

		Rankings			
	W(d(A+))		W(d(A-))		
	α1		α2		
(1;0)	(0,75;0,25)	(0,5;0,5)	(0,25;0,75)	(0;1)	Mahalan.
5	5	5	6	6	A1
4	4	4	3	2	A2
1	1	2	4	4	A3
3	3	3	2	2	A4
6	6	6	5	5	A5
2	2	1	1	1	A6

(1;0)	(0,75;0,25)	(0,5;0,5)	(0,25;0,75)	(0;1)	Euclidea
5	5	6	6	6	A1
3	3	3	2	2	A2
1	1	2	4	4	A3
3	3	3	2	2	A4
6	6	5	5	5	A5
2	2	1	1	1	A6

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GRUPO DECISIÓN MULTICRITERIO ZARAGOZA