Matching markets and interval order preferences: strategy-proofness vs efficiency

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Gale-Shapley college admission problem

Players

- Set of applicants A, set of universities B
- Each applicant $a \in A$ can be admitted to at most one university
- Each university $b \in B$ can admit no more than q_b students.
- Each player has linear order preferences over players on the other side

Definition

Matching μ is a mapping from $A \cup B$ to subsets of $A \cup B$ such that:

1. $a \in \mu(b) \Leftrightarrow b = \mu(a)$
2. $\forall a \in A \ \mu(a) = \{b\} \ (b \in B)$ or $\mu(a) = a$
3. $\forall b \in B \ \mu(b) \subseteq A$ or $\mu(b) = b$
4. $\forall b \in B \ |\mu(b)| \leq q_b$

Stability of matching

Matching is called stable if the following properties hold (Gale, Shapley, 1962):

- individual rationality of students: no student is matched to an unacceptable university,
- individual rationality of universities: no university admits an unacceptable student,
- "empty seats" stability: no university-student pair exists such that an applicant prefers this university to her current match and the university finds applicant acceptable and has empty seats
- pairwise stability: no university-student pair exists such that an applicant prefers this university to her current match and the university prefers this student to at least one of currently admitted students.

Deferred acceptance procedure

Gale and Shapley provided a constructive proof of existence of a stable matching.

Deferred acceptance procedure (students proposing)

- Step 1. Each student applies to her most preferred university. Each university temporary admits no more than q_b best students and rejects the others.
- ...
- Step k. Each rejected student applies to her second most preferred university. Each universiy consideres all currently applying students (both remaining from the previous steps and applied at the k-th), temporary admits no more than q_b among them and rejects the others.

When each student is temporary assgined to university or is rejected by all acceptable universities, procedure stops.

Properties of DA procedure

- DA procedure always produces a stable matching, call it μ_A,
- For students μ_A is a unique Pareto-optimal stable matching,
- Reporting preferences truthfully is a weakly-dominant strategy for each student.

Unfortunately, when preferences of agents are **not linear orders**:

- DA procedure is not well-defined in case of ties
- Even If DA procedure is redefined, efficiency do not necessary hold

The Model

Players

- Each student $a \in A$ can be admitted to one university
- Each university $b \in B$ can admit no more than q_b students.

Preferences

- R - the profile of students' preferences over universities. $\forall a \in A$ R_a is a linear order on $B \cup a$.
- $\succeq$ - the profile of universities' preferences over students. $\forall b \in B$ $\succeq_b$ is an interval order on $A \cup b$.
- $\forall b$ $\succeq_b$ satisfies "no indifference with empty set" property, i.e. $\forall b \in B, \forall a \in A$ either $a \succ_b \varnothing$ or $a \prec_b \varnothing$.
- preference relation of each university over the sets of applicants satisfies the responsiveness to the preference relation over individuals: $\forall b \in B$, $\forall a_1, a_2 \in A$, $A' \subset A$ if $a_1 \succ_b a_2$ then $A' \cup a_1 \succ_b A' \cup a_2$.

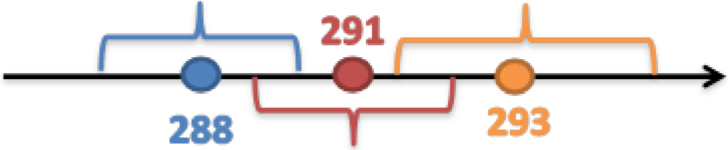
The interval order preferences: motivating example

Three students apply to a university b

- a_1's exam score is 293
- a_2's exam score is 291
- a_3's exam score is 288

University b has the following preferences:

$$a_1 \text{ is indifferent to } a_2,$$
$$a_2 \text{ is indifferent to } a_3,$$
$$\text{but } a_1 \text{ is preffered to } a_3.$$



This is an example of a natural situation, where negative transitivity property for preference relation does not hold.

The interval order preference relation

Interval order

Binary relation $\succeq$ on the set A is an **interval order**, if there exists an interval function $I : A \to R^2$, such that for each element in A it chooses an interval $[l, u]$, such that $a_1 \succ a_2$ if and only if $l(a_1) > u(a_2)$.

Semiorder

Semiorder $\succeq$ is an interval order with an interval function such that $(u_a - l_a) = const$. In other words, in case of a semiorder, the length of the 'error' interval does not depend on the alternative.

Simplest semiorder

Simplest semiorder is a semiorder such that $\forall a_1, a_2$ there exists no more than one a_3 such that negative transitivity condition does not hold.

Existence of stable matching

Theorem

Stable matching always exists in case, when universities have interval order preferences over students

Theorem

For each stable matching μ there exists such universities' preference profile $\succ$, which consists of linear order preferences and doesn't contradict original profile $\succeq$, such that matching μ is also stable under this strict preference profile.

Applicant Pareto-efficient stable matching

Erdil and Ergin, 2006, propose a procedure which allows to find student Pareto-optimal stable matching in case of universities' weak order preferences.
We modify it for the case of the interval order preferences and prove, that it works in our model.

1. Preferences of universities are arbitrary transformed to linear orders and Deferred Acceptance procedure with students proposing is applied. Result is a matching μ, stable but not necessarily Pareto-efficient.
2. We try to find so-called Stable Improvement Cycle. If SIC exists, then we can improve μ.
3. Procedure ends, when we arrive to the matching μ' which does not have Stable Improvement Cycle.

Stable Improvement Cycle

$$C(b,\mu) = \{a : bR_a\mu(a)\},\ D(b,\mu) = \{a \in C : \forall a' \in C a \succeq_b a'\}.$$

Definition

Stable Improvement Cycle consists of distinct applicants $a_1, ..., a_n \equiv a_0$ $(n \geq 2)$ *such that*

- $\mu(a_i) \in B$ *(each student in a cycle is assigned to a university),*
- $\forall a_i\ \mu(a_{i+1}) R_{a_i} \mu(a_i)$
- $\forall a_i\ a_i \in D(\mu(a_{i+1}), \mu)$*($a_i$ is one of the best students among those who prefer $\mu(a_{i+1})$ to her current match)*

Applicant Pareto-efficiency: necessary and sufficient condition

Theorem

Fix $\succeq$ and R, and let μ be a stable matching. If μ is student-side Pareto-dominated by another stable matching, the it admits a Stable Improvement Cycle.

Applicant Pareto-efficient mechanism

1. Construct linear extensions of interval order preference relations
2. Apply DA procedure with transformed preference profile
3. Search for a Stable Improvement Cycle. If cycle is found, improve a matching by exchanging seats among students in the SIC.
4. Search for a SIC until Pareto-efficient matching is reached.

Properties of the mechanism

- Applicant Pareto-efficient stable matching is always found
- Mechanism is not strategy-proof both for students and for universities

Proposition (Abdulkadiroglu, Pathak, Roth, 2005)

There does not exist an applicant strategy-proof mechanism, which would produce a matching, that Pareto-dominates the result of DA procedure with some tie-breaking rule.

Tie-breaking: example

Now we will check, how 'unfortunate' tie-breaking leads to an inefficient stable matching under applicant-proposing DA procedure

$P(a_1) : b_1 \succ b_2 \succ b_3 \quad \succ(b_1) : a_3 \approx a_2 \approx a_1, a_3 \succ a_1$
$P(a_2) : b_2 \succ b_1 \succ b_3 \quad \succ(b_2) : a_1 \approx a_3 \approx a_2, a_1 \succ a_2$
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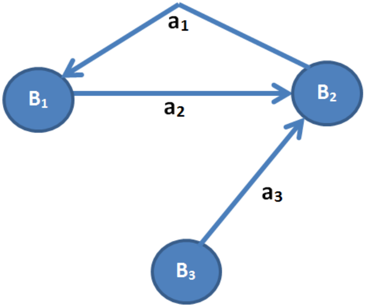
Procedure is finished, a stable matching is constructed!

Tie-breaking: example

Unfortunately, the matching is not efficient for applicants. There exists a Stable Improvement Cycle.



Preferences

$P(a_1) : b_1 \succ \mathbf{b_2} \succ b_3 \quad \succ (b_1) : a_3 \approx a_2 \approx a_1, a_3 \succ a_1$
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Mechanism with lower chances of an inefficient outcome

- Fix set of applicants A and set of universities B.
- Consider some particular university b.
- Suppose that preferences of all other applicants and universities are chosen randomly from uniform distribution.

Lemma 1

Fix preference relation of university b. Under student-proposing DA-procedure specific tie-breaking rule does not affect the probability of receiving proposition from each particular applicant.

Therefore, when we compare different tie-breaking rules, only the number of possible edges in an improvement graph matters.

Mechanism with lower chances of an inefficient outcome

'Reversal' tie-breaking rule

- Step 1. Under preference relation $\succ_b$ find a maximal set of undominated alternatives $I_1 \in A$. Let $|I_1| = k$.
- Step 1.0 Find an alternative (we'll call it a_k) with the lowest $I(a_k)$. Let $a_k \succ'_b a_i$ for each a_i in I_1.
- Step 1.i Choose an alternative (a_{k-i}) with the lowest $I_{a_{k-i}}$ among remaining and let this alternatives all other alternatives, remaining in the I_1.
- At the end of step 1 we will get $a_k \succ' a_{k-1} \succ' ... \succ' a_1$.
- Step 2. Under preference relation $\succ_b$ find a maximal set of undominated alternatives $I_2 \in A \setminus I_1$.

Mechanism with lower chances of an inefficient outcome

Definition

Regular semiorder is a semiorder preference relation, where each maximal set of incomparable alternatives (anti-chain) has the same cardinality.

Theorem

If university preference relation is a regular semiorder, then chances of forming an inefficient matching are the lowest with 'reversal' tie-breaking rule

Mechanism with lower chances of an inefficient outcome

New stable mechanism

1. Break ties in preferences according to the rule described above
2. Apply student-proposing DA procedure

Mechanism properties

- As it uses a tie-breaking rule before DA procedure, it is strategy-proof for students
- Outcome may be an Applicant-inefficient matching, but probability is lower then for GS with random tie-breaking rule.

Thank you!