Sensitivity of linear convolution from expert judgments

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Linear convolution of particular criteria is often used as aggregated criterion in problems of multicriteria optimization. The weights of significance of partial criteria can be obtained on the basis of their ranking by experts. Mistake in expert judgments may lead to inaccuracy in determining the weights and (as a consequence) to the inaccuracy in the solution of the problem of multicriteria optimization.

The aim of this work is to determine the influence of deviations of expert judgment in the solution proccess of multicriteria optimization problem.

Low sensitivity would reduce the costs of expertise.

1. ***Problem statement.***

Here we assume the existence of n nonlinear criteria

$$f_i(x)= f_i(x_1, x_2, ..., x_m), i=1,2,\dots,n$$

and we consider the multicriteria optimization problem

$$f_i(x) \to \max_{x\in X}, i=1,\dots,n$$

$X \subseteq R^m$ is open set. We assume that functions f_i strictly concave and sufficiently smooth functions.

Now we introduce the linear convolution of our criteria

$$F^0(x)= a^0{}_1 f_1+ a^0{}_2 f_2+\dots+ a^0{}_n f_n \to max \qquad (1)$$

where $a^0{}_1, a^0{}_2, \dots, a^0{}_n \geq 0$, $a^0{}_1+a^0{}_2+\dots + a^0{}_n =1$. The weights coefficients: $a^0{}_1, a^0{}_2, \dots, a^0{}_n$ reflect the relative importance of the criteria and are based on expert judgments. So the change of expert judgments leads to the other weights $a_1, a_2, \dots, a_n$ and its representations possible as

$$a_i=a^0{}_i+ \varepsilon\, a^1{}_i, \quad 0<\varepsilon << 1.$$

Let

$$< X^{0*}=(x^{0*}{}_1, x^{0*}{}_2, \dots, x^{0*}{}_m), F(X^{0*}) > \quad and \quad < X^*=(x^*{}_1, x^*{}_2, \dots, x^*{}_m), F(X^*) >$$

- are two the optimal solutions of (1) before and after the change of expert judgments.

Under the task of the sensitivity solution determination of multicriteria optimization problem we mean the finding deviations of the optimization problem solution (1) as the consequence of deviations in some metrics of expert judgments

$$<\delta X^*,\delta F(X^*)> = <X^*, F(X^*)> - <X^{0*},F^0(X^{0*}) > \qquad (2)$$

If the sensitivity of the solution (in some norm) is not higher then predetermined value, it indicates the absence of the need for clarification of expert assessments.

2. Sensitivity problem expert ranking.

The resulting ranking of partial criteria is a set of ranks $\boldsymbol{r_1, r_2, \ldots , r_n}$ – some set of integer positive numbers. In practice, a few ways to determine the criteria weights based on the rankings, the simplest of which is to use the formula

$$a_i = 2(n+1- r_i)/(n^2+n) \qquad (3)$$

in which the criteria weight is the ratio of the corresponding rank descending significance $(n+1- r_i)$ to the sum of the ranks of all criteria $(n^2+n)/2$.

Consider t groups of criteria in order of increasing rank: $G_1, G_1, \ldots, G_t$. Group G_j are non-empty, although some of them may consist of only one element.

Expert judgment - group ordering criteria by significance. Elementary change in judgment that is the transfer of some criteria from one criteria group to the neighboring. This changes the ranks of all the criteria of neighboring groups.

It is easy to see that

- such a change is simple and evident(and elementary in this sense);
- any change in the group ordering criteria can be represented as a superposition of elementary changes.

Let us estimate the rank deviations of partial criteria. It is easy show that a change of the significance criterion f_k in larger (lesser) side leads to slight changes in the weight coefficients (see the next Tables).

Old values of ranks and weights

Criteria	f_{k-s}	f_{k-s+1}	…	f_{k-1}	f_k	f_{k+1}	…	f_{k+p}
Ranks	***(2k-s)/2***	***(2k-s)/2***	…	***(2k-s)/2***	***(2k-s)/2***	***(2k+p+1)/2***	…	***(2k+p+1)/2***
Weights	***2(n+1-((2k-s)/2)/(n²+n)***					***2(n+1-((2k+p+1)/2)/(n²+n)***		

New values of ranks and weights

Критерии	f_{k-s}	f_{k-s+1}	…	f_{k-1}	f_k	f_{k+1}	…	f_{k+p}
Ranks	***(2k-s-1)/2***	***(2k-s-1)/2***	…	***(2k-s-1)/2***	***(2k+p)/2***	***(2k+p)/2***	…	***(2k+p)/2***
Weights	***2(n+1-((2k-s-1)/2)/(n²+n)***				***2(n+1-((2k+p)/2)/(n²+n)***			

Changes ranks and weights

Criteria	f_{k-s}	f_{k-s+1}	…	f_{k-1}	f_k	f_{k+1}	…	f_{k+p}
Changes of ranks	***- ½***	***- 1/2***	…	***- ½***	***(k+p)/2***	***- 1/2***	…	***- ½***
Changes of weights	***1/(n²+n)***				***- (p+s)/ (n²+n)***	***1/(n²+n)***		

Thus if after initial ranking operation the criterion f_k belongs to a group with s criteria $\{f_{k-1}, f_{k-2},\ldots, f_{k-s}\}$, having with him the same rank, and the change (increasing) rank of this criterion leads to its transition to the next group with p criteria $\{f_{k+1}, f_{k+2},\ldots, f_{k+p}\}$, having other rank.

So iIn this case

- the weights of the first and second groups of criteria to increase by a value of $1/(n^2+n)$;
- weight of criterion f_k decreases by an amount $(p+s)/(n^2+n)$;
- the other weights remain unchanged;
- criteria ranks first and second groups will decrease by 0.5;
- rank criterion f_k increase the amount of $(p + s) / 2$;
- ranks of other criteria will not change.

Transition criterion in the previous group would give opposite results.

3. Estimation of sensitivity of solving the optimization problem of expert judgments on the basis of a small parameter.

We estimate the impact of changes in elementary expert judgments on the deviation of the optimization problem. We believe the total number of criteria n sufficiently large(from practice point of view here we may take $n>2$), so the value $\varepsilon = 1 / (n^2 + n)$ can be used as a small parameter. Then our linear convolution (1) and the corresponding optimization problem can be represented as

$$F = F^0 + \varepsilon F^1 \rightarrow max, \quad (4)$$

$$F^1 = (p+s-1)f_k + f_{k-s} + f_{k-s+1} + \ldots + f_{k+p}$$

where the term F^1 generated by a change in criteria weights.

The resulting optimization problem can be viewed as a special case of a more general perturbed optimization problem.

The solution X^* in (4) depends on a small positive parameter $0 < \varepsilon << 1$. This allows on the basis of so-called direct scheme, using a expansion of the solution X^* and the function F in (4) in regular series by ε. After that we may to construct a series of maximization problems for the terms of the expansion of X^*.

So, let the solution of the perturbed optimization problem (4) has the form

$$X^*(\varepsilon) = X^{*0} + \varepsilon X^{*1} + \varepsilon^2 X^{*2} + \ldots \quad (5)$$

Then, substituting (5) into (4) and expand $\boldsymbol{F}$ in powers of ε, we obtain two optimization problems

1) initial problem (1) - the problem for $\boldsymbol{X^{*0}}$ -

$$\boldsymbol{X^{*0} = arg\ max\ \{F^0(X^0)\}} \tag{6}$$

2) the problem for $\boldsymbol{X^{*1}}$ -

$$\boldsymbol{X^{*1} = arg\ max\{(X^{*1})^T \nabla^2 F^0(X^{*(0)})\ X^{*1} + (\nabla F^1(X^{*0}))^T\ X^{*1}\}} \tag{7}$$

where $\boldsymbol{\nabla^2 F^0(X^{*(0)})}$ - hessian function $\boldsymbol{F^0}$ and $\boldsymbol{\nabla F^1(X^{*(0)})}$ - gradient of the function $\boldsymbol{F^1}$, taken at the solution of initial problem (1) or (6) solution $\boldsymbol{X^{*(0)}}$.

After solution (6) and (7) we may obtain the sensitivity by solution

$$\boldsymbol{\delta X \approx \varepsilon X^{*1} = -\ \varepsilon\ (\nabla^2 F^0(X^{*0}))^{-1}\ \nabla F^1(X^{*0})} \tag{8}$$

and corresponding sensitivity by linear convolution value.

Example.

$f_1 = -(x_1)^2-(x_2)^2$, $f_2 = -(x_1-1)^2-(x_2-1)^2$, $f_3 = -(x_1-2)^2-(x_2-2)^2$,

$f_4 = -(x_1-4)^2-(x_2-4)^2$

Let

$$f_1 > f_2 > f_3 \sim f_4 .$$

After elementary change of expert judgment we have

$$f_1 > f_2 \sim f_3 > f_4 .$$

For investigation sensitivity optimal problem solution we have the given upper limit – 5%.

Our ranks and criterion weights:

Before

f_i	f_1	f_2	f_3	f_4
r_i	1	2	3,5	3,5
a_i	0,4	0,3	0,15	0,15

After

f_i	f_1	f_2	f_3	f_4
r_i	1	2,5	2,5	4
a_i	0,4	0,25	0,25	0,1

Linear convolution (before):

$F^0 = 0{,}4\, f_1 + 0{,}3\, f_2 + 0{,}15\, f_3 + 0{,}15\, f_4$

And after:

$$F = 0{,}4\, f_1 + 0{,}3\, f_2 + 0{,}15\, f_3 + 0{,}15\, f_4 + \varepsilon(-f_2 + 2\, f_3 - f_4)$$

Where $\varepsilon = 1/(n^2+n) = 1/(4^2+4) = 0{,}05$.

Solution before

$$F^0 = -x_1^2 + 2(0,3+0,15*2+0,15*4)x_1 - (0,3+0,15*4+0,15*16) -$$

$$-x_2^2 + 2(0,3+0,15*2+0,15*5)x_2 - (0,3+0,15*4+0,15*16) =$$

$$= -(x_1^2 - 2,4x_1 + 3,3) - (x_2^2 - 2,4x_2 + 3,3) \rightarrow \boldsymbol{max}$$

$x^{*(0)}_1 = 1.20$, $x^{*(0)}_2 = 1.20$, $F^{*(0)} = -3.72$.

Solution after

$$F = F^0 + \varepsilon F^1 = [-(x_1^2 - 2,4x_1 + 3,3) - (x_2^2 - 2,4x_2 + 3,3)]$$

$$+0,05[2x_1 + 2x_2 + 26] \rightarrow \boldsymbol{max}$$

$x_1 \approx 1,15$; $x_2 \approx 1,15$, $F \approx -3,59$.

Such the deviation by solution is near 4.2% by every components of ***X*** and near 4.5% by linear convolution value ($\boldsymbol{\delta F}$), that is less then given limit 5%.

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