Determination of Transmission Capacity For a Two-Node Market

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Model of a two-node market

- A_i – finite set of producers at the local market i, $i = 1, 2$
- $E^a(q)$ – cost function of producer a, $a \in A_i$
- $d_i(p)$ – demand function at the local market i, $i = 1, 2$
- k – loss coefficient
- C – transmission capacity

Strategy of producer a is a non-decreasing supply function $r^a(p)$ that determines the output volume depending on the price p.

Model of a two-node market

Clearing prices $\overline{p}_i$ for isolated markets are determined by the equations $\sum_{a\in A_i} r^a(\overline{p}_i) = d_i(\overline{p}_i)$, $i = 1, 2$.
If

$$1 - k \leq \overline{p}_2/\overline{p}_1 \leq (1 - k)^{-1}, \quad (1)$$

then there is no transmission from one market to the other and the nodal prices are equal to the prices of isolated markets.
Otherwise let $\overline{p}_2/\overline{p}_1 > (1 - k)^{-1}$. In this case, the network administrator determines the volume of the good v that will be transmitted from the first market to the second market.

Nodal prices and the flow

Nodal prices $p_1(v)$ and $p_2(v)$ and the flow v are determined by the system:

$$\begin{cases} \sum_{a \in A_1} r^a(p_1) = d_1(p_1) + v \\ \sum_{a \in A_2} r^a(p_2) = d_2(p_2) - (1-k)v \\ \left[\begin{array}{l} \begin{cases} p_1(v) = (1-k)p_2(v) \\ v < C \end{cases} \\ \begin{cases} p_1(v) \leq (1-k)p_2(v) \\ v = C \end{cases} \end{array} \right. \end{cases}$$

The optimal strategy under perfect competition:

$$s^a(p) \stackrel{\text{def}}{=} \operatorname*{Argmax}_{q^a}(q^a p - E^a(q^a)),\ s_i(p) \stackrel{\text{def}}{=} \sum_{a \in A_i} s^a(p),\ i = 1, 2.$$

$\widetilde{p}_i(C)$, $i = 1, 2$ – nodal prices corresponding to Walrasian supply functions depending on the transmission capacity.

Prices $\widetilde{p}_i(0)$ meet the equations $d_i(\widetilde{p}_i) \in s_i(\widetilde{p}_i)$, $i = 1, 2$.

If there is a flow from the first market to the second market, the prices $\widetilde{p}_1(C)$ и $\widetilde{p}_2(C)$ satisfy the following conditions:

$$\begin{cases} s_1(\widetilde{p}_1) = d_1(\widetilde{p}_1) + v \\ s_2(\widetilde{p}_2) = d_2(\widetilde{p}_2) - (1-k)v \\ \left[\begin{array}{l} \begin{cases} \widetilde{p}_1 = (1-k)\widetilde{p}_2 \\ v < C \end{cases} \\ \begin{cases} \widetilde{p}_1 \le (1-k)\widetilde{p}_2 \\ v = C \end{cases} \end{array} \right. \end{cases} \tag{2}$$

Let functions $\widetilde{p_1}^0(v)$ and $\widetilde{p_2}^0(v)$ be implicitly determined by the first and the second equations of the system (2) respectively. Assume that $\widetilde{p_1}^0(0) < (1-k)\widetilde{p_2}^0(0)$.

Theorem 1

There exists a value of the transmission capacity $\widehat{C}$ determined by the condition $\widetilde{p_1}^0(\widehat{C}) = (1-k)\widetilde{p_2}^0(\widehat{C})$ such that if $C < \widehat{C}$, then at the equilibrium

$$v = C, \quad \widetilde{p_i}(C) = \widetilde{p_i}^0(C), \quad i = 1, 2, \tag{3}$$

$$\widetilde{p_1}(C) < (1-k)\widetilde{p_2}(C). \tag{4}$$

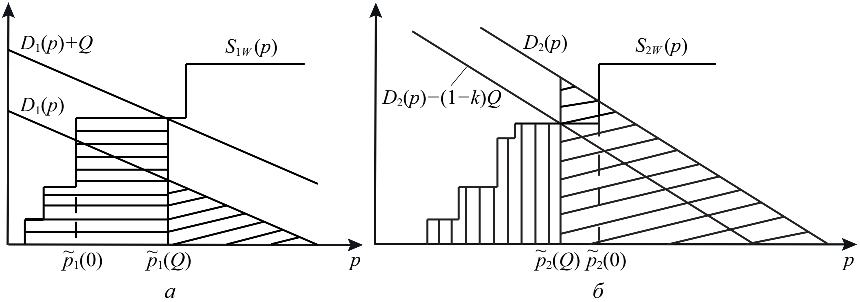
If $C > \widehat{C}$, then

$$v = \widehat{C} < C, \quad \widetilde{p_i}(C) = \widetilde{p_i}^0(\widehat{C}), \quad i = 1, 2. \tag{5}$$

$$N(C) = \mathrm{P}_1(C) + \mathrm{P}_2(C) + \mathrm{S}_1(C) + \mathrm{S}_2(C) + \mathrm{T}(C), \text{where}$$

- $\mathrm{P}_i(C) = \sum_{a \in A_i} \left(\widetilde{p}_i(C) s^a(\widetilde{p}_i(C)) - E^a(s^a(\widetilde{p}_i(C))) \right) =$
  $= \int_0^{\widetilde{p}_i(C)} s_i(p)dp$ – producers' profit at market i,
- $\mathrm{S}_i(C) = \int\limits_{\widetilde{p}_i(C)}^{\infty} d_i(p)dp$ – consumer surplus at market i,
- $\mathrm{T}(C) = \widetilde{p}_2(C)(1-k)C - \widetilde{p}_1(C)C$ – the benefit of the network system.

Producers' profit and consumer surplus



The costs of the transmission line construction:

$$B(C) = \begin{cases} 0, & \text{if } C = 0, \\ b_f + b_v(Q), & \text{if } C > 0, \end{cases}$$

where $b_v(C)$ is a convex and increasing function that determines variable costs, $b_v(0) = 0$; b_f is constant costs.
Taking into account the construction costs, the total welfare is $W(C) = N(C) - B(C)$.

Theorem 2

Function $N(C)$ is concave and increases in C if $C \leq \widehat{C}$. In addition, $N^{'}(C) = (1-k)\widetilde{p}_2(C) - \widetilde{p}_1(C)$.

Theorem 3

The optimal transmission capacity C^* equals zero if $(1-k)\widetilde{p}_2(0) - \widetilde{p}_1(0) \leq b_v'(0)$. If this inequality does not hold, the value C^{*L} corresponding to a local maximum is determined by the equation $(1-k)\widetilde{p}_2(C^{*L}) - \widetilde{p}_1(C^{*L}) = b_v'(C^{*L})$ and satisfies $C^{*L} < \widehat{C}$. If $W(C^{*L}) > W(0)$ then $C^* = C^{*L}$. Otherwise $C^* = 0$.

The flow of good between markets affects the benefit of transmission system, consumer surplus and producers' profit as follows:

The first market:	$\Delta \mathrm{P}_1 > 0$, $\Delta \mathrm{S}_1 < 0$.
The second market:	$\Delta \mathrm{P}_2 < 0$, $\Delta \mathrm{S}_2 > 0$.
Profit of the network system:	$\widetilde{p}_2(C^*)(1-k)C^* - \widetilde{p}_1(C^*)C^*$.

A strategy of producer a is a production volume $q^a \in [0, V^a]$. Let $\overrightarrow{q_i} = (q^a, a \in A_i)$ be a strategy profile for the node $i = 1, 2$. For the separated markets, the prices p_i^{*0}, $i = 1, 2$, are
$p_i^{*0}(\overrightarrow{q_i}) = d_i^{-1}(\sum_{a \in A_i} q^a)$, $i = 1, 2$.
Nodal prices $p_1(v)$ and $p_2(v)$ and the flow v are determined by the system:

$$\begin{cases} \sum\limits_{a \in A_1} q^a = d_1(p_1) + v \\ \sum\limits_{a \in A_2} q^a = d_2(p_2) - (1-k)v \\ \left[\begin{array}{l} \begin{cases} p_1 = (1-k)p_2 \\ v < C \end{cases} \\ \begin{cases} p_1 \le (1-k)p_2 \\ v = C \end{cases} \end{array} \right. \end{cases} \tag{6}$$

Transmission of the good is unprofitable since the prices for the separated markets meet conditions $\lambda^{-1} < p_2^*/p_1^* < \lambda$, where $\lambda = (1-k)^{-1}$.

The first order conditions (FOCs) for such equilibrium are:

$$q^{a*} \in (p_i^* - E^{a'}(q^{a*}))|d_i'(p_i^*)| \text{ , for every } a \in A_i$$

$$\text{such that } E^{a'}(0) < p_i^*, i = 1, 2,$$

$$q^{a*} = 0 \text{ if } E^{a'}(0) \geq p_i^*,$$

where $E^{a'}(q) = [E_-^{a'}(q), E_+^{a'}(q)]$ at the jump points of the marginal cost function.

The equilibrium prices p_i^* are determined by the equations

$$\sum_{a \in A_i} s_{iC}^a(p_i^*) = d_i(p_i^*), \quad i = 1, 2$$

.

Type B_{1-2} equilibrium (1)

At the type B_{12} equilibrium, $v \in (0, C)$ and $\lambda p_1^* = p_2^*$.
Under small variations of the price, the demand at the first market is

$$d_1(p_1) + \lambda(d_2(\lambda p_1) - \sum_{a \in A_2} q^a).$$

Thus the price p_1^b meets the equation

$$\sum_{a \in A_1} q^a = d_1(p_1^b) + \lambda(d_2(\lambda p_1^b) - \sum_{a \in A_2} q^a).$$

The FOCs for this type of equilibrium are: for every $a \in A_1$

$$q^{a*} \in (p_1^{*b} - E^{a'}(q^{a*}))|d_1'(p_1^{*b}) + \lambda^2 d_2'(\lambda p_1^{*b})| \text{ if } E^{a'}(0) < p_1^{*b},$$

$$q^{a*} = 0 \text{ if } E^{a'}(0) \geq p_1^{*b}.$$

The demand for producers at the second market is

$$d_2(\lambda p_1) + 1/\lambda(d_1(p_1) - \sum_{a \in A_1} q^a),$$

and the FOCs for the Nash equilibrium are

$$q^{a*} \in (\lambda p_1^{*b} - E^{a'}(q^{a*}))|d_2{}'(\lambda p_1^{*b}) + d_1{}'(p_1^{*b})/\lambda^2| \text{ if } E^{a'}(0) < p_2^{*b},$$

$$q^{a*} = 0 \text{ if } E^{a'}(0) \geq p_2^{*b}.$$

Type C_{1-2} equilibrium

At the c_{12} type equilibrium, $v = C$ and $\lambda p_1^* < p_2^*$.
The FOCs :

$$q^{a*} \in (p_i^{*c} - E^{a'}(q^{a*}))|d_i{}'(p_i^{*c})| \quad \text{if } E^{a'}(0) < p_i^{*c}, i = 1, 2;$$

$$q^{a*} = 0 \quad \text{if } E^{a'}(0) \geq p_i^{*c}.$$

The total supply at each market balances the demand:

$$\sum_{a \in A_1} q^{a*} = d_1(p_1^{*c}) + C,$$

$$\sum_{a \in A_2} q^{a*} = d_2(p_2^{*c}) - \lambda^{-1} C.$$

At the type d_{12} equilibrium, $v = C$ and $\lambda p_1^* = p_2^*$. The FOCs for producers at the first node are

$$(p_1^* - E_-^{a'}(q^{a*}))|d_1'(p_1^*) + \lambda^2 d_2'(\lambda p_1^*)| \geq q^{a*} \geq (p_1^* - E_+^{a'}(q^{a*}))|d_1'(p_1^*)|.$$

The FOCs for the second node are

$$(\lambda p_1^* - E_-^{a'}(q^{a*}))|d_2'(\lambda p_1^*)| \geq q^{a*} \geq (\lambda p_1^* - E_+^{a'}(q^{a*}))|d_2'(\lambda p_1^*) + d_1'(p_1^*)/\lambda^2|.$$

The total supply at each market balances the demand:

$$\sum_{a \in A_1} q^{a*} = d_1(p_1^{*c}) + C,$$

$$\sum_{a \in A_2} q^{a*} = d_2(p_2^{*c}) - \lambda^{-1} C.$$

Cournot equilibrium depending on the transmission capacity C (1)

- Cournot prices p_i^{*0}, $i = 1, 2$ for isolated markets are determined by the equations: $s_{iC}(p_i^{*0}) = d_i(p_i^{*0})$, $i = 1, 2$.
- $\Delta_{ij}^1(\lambda, p) \stackrel{\text{def}}{=} s_{1Ci-j}(\lambda, p) - d_1(p)$
  $\Delta_{ij}^2(\lambda, p) \stackrel{\text{def}}{=} s_{2Ci-j}(\lambda, p) - d_2(p)$.

Consider the case $\lambda = 1$:

- $\Delta_{ij}^1(1, p) = \Delta_{ji}^1(1, p) \stackrel{\text{def}}{=} \Delta^1(p)$;
  $\Delta_{ij}^2(1, p) = \Delta_{ji}^2(1, p) \stackrel{\text{def}}{=} \Delta^2(p)$
- Let prices $\overline{p}_1$ и $\overline{p}_2$ be determined by the conditions: $\Delta^i(\overline{p}_i) = 0$, $i = 1, 2$

Cournot equilibrium depending on the transmission capacity C (2)

Theorem 4

Let $d_i(p) > 0$ and $d_i'(p)$ be non-increasing if $p \in (0, M_i)$; $d_i(p) = 0$ if $p \geq M_i$, $i = 1, 2$, and p_1^{*0}, p_2^{*0}, $\overline{p}_1$, $\overline{p}_2$, M_1, M_2 meet conditions $p_1^{*0} < p_2^{*0} < M_2 < M_1$, $\overline{p}_1 < \overline{p}_2$. Then for any $C > 0$, there exists at most one equilibrium for λ close enough to 1.
Moreover, there is a value $\underline{C} \in (0, \overline{C})$, where $\overline{C} = s_{1C1-2}(p_1^{*b}) - d_1(p_1^{*b})$, such that if $C \in (0, \underline{C})$ then there exists a C_{1-2} equilibrium; if $C > \overline{C}$, there exists a B_{1-2} equilibrium; if $C \in (\underline{C}, \overline{C})$, only D_{1-2} type equilibrium is possible.

Theorem 5

Let $d_i(p) = max\{\widehat{D_i} - d_i p, 0\}$, $i = 1, 2$ and marginal costs be piecewise constant. Then for $C < \underline{C}$ (type C equilibrium) there exist intervals $C_j < C < C_{j+1}$ such that the total welfare function $TW(C)$ is concave in each of these intervals.

Theorem 6

If $C > \overline{C}$ (type B equilibrium), the total welfare function $TW(C)$ decreases. The optimal transmission capacity $C^* \leq \overline{C}$.

Theorem 7

Let $d_i(p) = max\{\widehat{D_i} - d_i p, 0\}$, $i = 1, 2$ and marginal costs be piecewise constant. Then for $C \in (\underline{C}, \overline{C})$ (type D equilibrium) under perfect competition at the second market, there exist intervals $z_j < C < z_{j+1}$ such that the total welfare function $TW(C)$ is concave in each of these intervals.

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