Strongly Time-consistent Solutions for Two-stage Network Games

Hongwei Gao [1] Leon Petrosyan [2] Artem Sedakov [2]

[1]College of Mathematics
Qingdao University

[2]Department of Game Theory and Statistical Decisions
Saint Petersburg State University

June 3, 2014

Outline

1. The model
2. Cooperation in two-stage network game
3. Strongly time-consistent solution
4. Numerical example

The Model

$N = \{1, \ldots, n\}$: a finite set of players who can interact with each other.

g: a finite set of pairs $(i, j) \in N \times N$, or a network.

If $(i, j) \in g$, we say that there is a link connecting players i and j, and, therefore, communication of the players.

In our setting we suppose that all links are undirected, i.e. $(i, j) = (j, i)$.

For the simplicity denote (i, j) as ij.

Consider a two-stage problem:

- At the first stage each player chooses his partners—other players with whom he wants to form links. After choosing partners and establishing links, players, thereby, form a network.
- At the second stage having the network formed, each player chooses a control influencing his payoff.

First Stage: Network Formation

$M_i \subseteq N \setminus \{i\}$: the set of players whom player $i \in N$ can offer a mutual link.

$a_i \in \{0, \dots, n-1\}$: the maximal number of links which player i can offer.

Behavior of player $i \in N$ at the first stage is a profile $g_i = (g_{i1}, \dots, g_{in})$ which components are defined as:

$$g_{ij} = \begin{cases} 1, & \text{if player } i \text{ offers a link to } j \in M_i, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

subject to the constrain:

$$\begin{aligned} \sum_{j \in N} g_{ij} &\leqslant a_i, \\ g_{ii} &= 0, \ i \in N. \end{aligned} \tag{2}$$

$G_i = \{g_i : (1) - (2) \text{ are true}\}, \quad i \in N.$

Players choose their behaviors at the first stage simultaneously and independently from each other.
In particular, player $i \in N$ chooses $g_i \in G_i$, and as a result the behavior profile $(g_1, \dots, g_n)$ is formed.
Having the behavior profile $(g_1, \dots, g_n)$ formed, an undirected link $ij = ji$ is established in network g if and only if

$$g_{ij} = g_{ji} = 1,$$

i.e. g consists of mutual links which were offered only by both players.

Example

Let $N = \{1, 2, 3, 4\}$ and players choose the following behaviors at the first stage: $g_1 = (0, 1, 1, 1)$, $g_2 = (1, 0, 1, 0)$, $g_3 = (1, 1, 0, 0)$, $g_4 = (0, 0, 1, 0)$. The resulting network g contains three links $\{12, 13, 23\}$.

Second Stage: Controls

$N_i(g) = \{j \in N \setminus \{i\} : ij \in g\}$: neighbors of player i in network g.

Let $d_i(g) = (d_{i1}(g), \ldots, d_{in}(g))$ be defined as follows:

$$d_{ij}(g) = \begin{cases} 1, & \text{if } i \text{ does not break the link formed} \\ & \text{at the first stage} \\ & \text{with player } j \in N_i(g) \text{ in network } g, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

$D_i(g) = \{d_i(g) : (3) \text{ is true}\}$.

Profile $(d_1(g), \ldots, d_n(g))$ changes network g and forms a new network, denoted by g^d. In network g^d all links ij such that either $d_{ij}(g) = 0$ or $d_{ji}(g) = 0$ are removed.

At the second stage player $i \in N$ chooses control u_i from a finite set U_i.

Behavior of player $i \in N$ at the second stage is a pair $(d_i(g), u_i)$: it defines, on the one hand, links to be removed $d_i(g)$, and, on the other hand, control u_i.

Payoff function K_i of player i: it rules, depends on player's behavior at the second stage as well as behavior of his neighbors in network g^d.

$K_i(u_i, u_{N_i(g^d)})$, $i \in N$, is non-negative real-valued function defined on the set $U_i \times \prod_{j \in N_i(g^d)} U_j$ and satisfies the following property:

(P): for any two networks g and g' and player i if $|N_i(g)| \geqslant |N_i(g')|$, the inequality $K_i(u_i, u_{N_i(g)}) \geqslant K_i(u_i, u_{N_i(g')})$ holds for all $(u_i, u_{N_i(g)}) \in U_i \times \prod_{j \in N_i(g)} U_j$ and $(u_i, u_{N_i(g')}) \in U_i \times \prod_{j \in N_i(g')} U_j$.

Cooperation in Two-stage Network Game

We study the cooperative case and answer three main questions:

- What is a cooperative solution in the game?
- Can it be realized in the game?
- Is it strongly time-consistent?

Suppose now that players jointly choose their behaviors at both stages of the game. Acting as one player and choosing $g_i \in G_i$, $u_i \in U_i$, $i \in N$, the grand coalition, N, maximizes the value:

$$\sum_{i \in N} K_i(u_i, u_{N_i(g)}). \tag{4}$$

Let the maximum be attained when players' behavior profiles g_i^*, u_i^*, $i \in N$ are chosen, and profile $(g_1^*, \ldots, g_n^*)$ forms network g^*.

Proposition

$d_i^*(g^*) = g_i^*$ for all $i \in N$, i.e. players should not remove links from network g^*.

Let

$$\sum_{i \in N} K_i(u_i^*, u_{N_i(g^*)}^*) = \max_{g_i \in G_i, i \in N} \max_{u_i \in U_i, i \in N} \sum_{i \in N} K_i(u_i, u_{N_i(g)}).$$

To allocate the maximal sum of players' payoffs according to some imputation, we construct an auxiliary cooperative TU-game (N, V). The characteristic function V is constructed as follows.

Proposition

In the cooperative two-stage network game the superadditive characteristic function $V(\cdot)$ in the sense of von Neumann and Morgenstern is defined as:

$$\begin{aligned} V(N) &= \sum_{i \in N} K_i(u_i^*, u^*_{N_i(g^*)}), \\ V(S) &= \max_{g_i \in G_i, i \in S} \max_{u_i \in U_i, i \in S} \sum_{i \in S} K_i(u_i, u_{N_i(g) \cap S}), \\ V(\varnothing) &= 0. \end{aligned}$$

For a singleton $\{i\}$, its value is defined in the following way:

$$V(\{i\}) = \max_{u_i \in U_i} K_i(u_i).$$

An imputation in the cooperative two-stage network game is an n-dimensional profile $\xi = (\xi_1, \dots, \xi_n)$, satisfying $\sum_{i \in N} \xi_i = V(N)$ and $\xi_i \geqslant V(\{i\})$ for all $i \in N$. The set of all imputations in the game (N, V) we denote by $I(V)$.

A cooperative solution concept in the auxiliary cooperative TU-game (N, V) is a rule that uniquely assigns a subset $CSC(V) \subseteq I(V)$ to the game (N, V). For example, if the cooperative solution concept is the core $C(V)$, then

$$CSC(V) = C(V) = \left\{ \xi = (\xi_1, \dots, \xi_n) : \sum_{i \in S} \xi_i \geqslant V(S),\ S \subseteq N \right\}.$$

Suppose that at the beginning of the game players jointly decide to choose behavior profiles g_i^*, u_i^*, $i \in N$ and then allocate it according to a specified cooperative solution concept $CSC(V)$ which realizes the imputation $\xi = (\xi_1, \ldots, \xi_n)$. It means that in the cooperative two-stage network game player $i \in N$ should receive the amount of ξ_i as his payoff.

What will happen if after the first stage (after choosing the profiles $g_1^*, \ldots, g_n^*$) players recalculate the imputation according to the same cooperative solution concept?

Recalculated Imputation is different!

Let g^* be formed at the first stage. If players recalculate the imputation, one needs to construct a new cooperative game $(N, v(g^*))$, provided network g^* is fixed.

Proposition

The superadditive characteristic function $v(g^*, \cdot)$ in the sense of von Neumann and Morgenstern is defined as:

$$
\begin{aligned}
v(g^*, N) &= \sum_{i \in N} K_i(u_i^*, u^*_{N_i(g^*)}), \\
v(g^*, S) &= \max_{u_i \in U_i, i \in S} \sum_{i \in S} K_i(u_i, u_{N_i(g^*) \cap S}), \\
v(g^*, \varnothing) &= 0,
\end{aligned}
$$

For a singleton $\{i\}$, its value is defined in the following way:

$$v(g^*, \{i\}) = \max_{u_i \in U_i} K_i(u_i) = V(\{i\}),$$

and it does not depend on the network.

An imputation is an n-dimensional profile $\xi(g^*) = (\xi_1(g^*), \ldots, \xi_n(g^*))$, satisfying both the efficiency condition and the individual rationality condition:

$$\begin{aligned}\sum_{i \in N} \xi_i(g^*) &= v(g^*, N), \\ \xi_i(g^*) &\geqslant v(g^*, \{i\}), \ i \in N.\end{aligned}$$

The set of all imputations in the game $(N, v(g^*))$ we denote by $I(v(g^*))$.

A cooperative solution concept in the auxiliary cooperative TU-game $(N, v(g^*))$ with fixed network g^* is a rule that uniquely assigns a subset $CSC(v(g^*)) \subseteq I(v(g^*))$ to the game $(N, v(g^*))$. For example, if the cooperative solution concept is the core $C(v(g^*))$, then

$$C(v(g^*)) = \Bigg\{\xi(g^*) = (\xi_1(g^*), \ldots, \xi_n(g^*)) : \sum_{i \in S} \xi_i(g^*) \geqslant v(g^*, S), \ S \subseteq$$

Definition

An imputation $\xi \in CSC(V)$ is said to be time-consistent if there exists an imputation $\xi(g^*) \in CSC(v(g^*))$ such that the following equality holds for all players:

$$\xi_i = \xi_i(g^*), \quad i \in N. \tag{5}$$

A cooperative solution concept $CSC(V)$ is time consistent if any imputation $\xi \in CSC(V)$ is time-consistent.

Definition

Imputation distribution procedure for ξ in the cooperative two-stage network game is a matrix

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \vdots & \vdots \\ \beta_{n1} & \beta_{n2} \end{pmatrix},$$

where

$$\xi_i = \beta_{i1} + \beta_{i2}, \quad i \in N.$$

The payment scheme is applied: player $i \in N$ at the first stage of the game receives the payment β_{i1}, at the second stage of the game he receives the payment β_{i2}.

Definition

Imputation distribution procedure β for ξ is time-consistent if

$$\xi_i - \beta_{i1} = \xi_i(g^*), \quad \text{for all } i \in N.$$

It is obvious that time-consistent imputation distribution procedure for $\xi = (\xi_1, \ldots, \xi_n)$ in the cooperative two-stage network game can be defined as follows:

$$\begin{aligned} \beta_{i1} &= \xi_i - \xi_i(g^*), \\ \beta_{i2} &= \xi_i(g^*), \quad i \in N. \end{aligned} \tag{6}$$

In case of the cooperative solution concept $CSC(V)$ assigns multiple allocations (for example, the core), more strict property of imputation distribution procedure can be used—the strong time-consistency property.

Definition

An imputation $\xi \in CSC(V)$ is said to be strongly time-consistent if the following inclusion is satisfied:

$$CSC(v(g^*)) \subseteq CSC(V). \quad (7)$$

A cooperative solution concept $CSC(V)$ is strongly time-consistent if any imputation $\xi \in CSC(V)$ is strongly time-consistent.

Therefore, the core $C(V)$ is strongly time-consistent if $C(v(g^*)) \subseteq C(V)$. Unfortunately, for strongly time-consistent imputation distribution procedures it is impossible even to derive formulas similar to (6).

Proposition

In two-stage cooperative games the core $C(V)$ is time-consistent but not strongly time-consistent.

Definition

Imputation distribution procedure β for ξ is strongly time-consistent if

$$(\beta_{11}, \ldots, \beta_{n1}) \oplus CSC(v(g^*)) \subseteq CSC(V), \tag{8}$$

where $a \oplus A = \{a + a' : a' \in A, a \in R^n, A \subset R^n\}$.

Note, that strongly time-consistent imputation distribution procedure β for an imputation from the core $C(V)$ satisfies the inclusion:

$$(\beta_{11}, \ldots, \beta_{n1}) \oplus C(v(g^*)) \subseteq C(V). \tag{9}$$

Numerical Example

$N = \{1, 2, 3\}$.

Subsets of players to whom each player can offer a link are: $M_1 = \{2, 3\}, M_2 = \{1, 3\}, M_3 = \{1\}$.

A number of links each players can offer: $a_1 = a_2 = a_3 = 1$.

Therefore, at the first stage sets of players' behaviors are:
$G_1 = \{(0, 0, 0); (0, 1, 0); (0, 0, 1)\}$,
$G_2 = \{(0, 0, 0); (1, 0, 0); (0, 0, 1)\}$,
$G_3 = \{(0, 0, 0); (1, 0, 0)\}$,
and only three networks can be formed at the first stage of the game: the empty network (the network without links, $g = \varnothing$), $g = \{12\}$, and $g = \{13\}$.

Suppose that sets of controls U_i at the second stage for any network g, realized at the first stage, are the same $U_1 = U_2 = U_3 = \{A, B\}$, and payoff functions are defined as:

$$
\begin{array}{lllll}
K_i(u_i): & K_i(A) = 1, & K_i(B) = 0, & i = 1, 2, 3, & \\
K_1(u_1, u_2): & K_1(A, A) = 2, & K_1(A, B) = 4, & K_1(B, A) = 1, & K_1(B, B) = 3, \\
K_1(u_1, u_3): & K_1(A, A) = 3, & K_1(A, B) = 5, & K_1(B, A) = 1, & K_1(B, B) = 3, \\
K_2(u_2, u_1): & K_2(A, A) = 2, & K_2(A, B) = 4, & K_2(B, A) = 1, & K_2(B, B) = 3, \\
K_3(u_3, u_1): & K_3(A, A) = 2, & K_3(A, B) = 5, & K_3(B, A) = 1, & K_3(B, B) = 4.
\end{array}
$$

Consider the case of cooperation at both stages. In this case the maximal value

$$\sum_{i \in N} K_i(u_i^*, u_{N_i(g^*)}^*) \quad = \quad 8,$$

and it can be reached if players choose the following behaviors:

$$\begin{aligned} g_1^* &= (0,0,1),\ g_2^* = (0,0,0),\ g_3^* = (1,0,0), \\ u_1^* &= B,\ u_2^* = A,\ u_3^* = B. \end{aligned}$$

Note that behavior profile g_1^*, g_2^*, g_3^* at the first stage forms the network $g^* = \{13\}$.

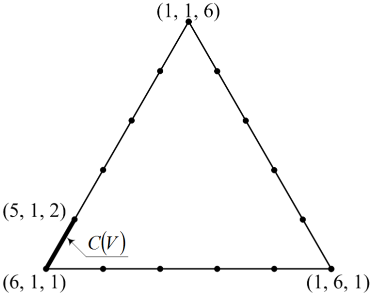
Calculate values of characteristic function $V(S)$ for all coalitions $S \subseteq N$: $V(N) = 8$, $V(\{1,2\}) = 6$, $V(\{1,3\}) = 7$, $V(\{2,3\}) = 2$, $V(\{1\}) = V(\{2\}) = V(\{3\}) = 1$.

The core $C(V)$:

$$\begin{aligned} \xi_1 + \xi_3 &= 7, \\ \xi_1 &\geqslant 5, \\ \xi_2 &= 1, \\ \xi_3 &\geqslant 1. \end{aligned}$$

Consider now cooperation at the second stage of the game, provided that network $g^* = \{13\}$ at the first stage is fixed.
Calculate values of characteristic function $v(g^*, S)$ for all coalitions $S \subseteq N$: $v(\{13\}, N) = 8$, $v(\{13\}, \{1,2\}) = v(\{13\}, \{2,3\}) = 2$, $v(\{13\}, \{1,3\}) = 7$, $v(\{13\}, \{1\}) = v(\{13\}, \{2\}) = v(\{13\}, \{3\}) = 1$.
The core $C(v(\{13\}))$:

$$\begin{aligned}
\xi_1(\{13\}) + \xi_3(\{13\}) &= 7, \\
\xi_1(\{13\}) &\geqslant 1, \\
\xi_2(\{13\}) &= 1, \\
\xi_3(\{13\}) &\geqslant 1.
\end{aligned}$$



Since $C(V) \subset C(v(g^*))$, the core $C(V)$ is time-consistent cooperative solution concept in two-stage network games but it is obvious that the core $C(V)$ is not strongly time-consistent (inclusion (7) does not hold).

Consider another cooperative solution concept—the τ-value. In the cooperative two-stage network game the τ-value $\tau = (\tau_1, \ldots, \tau_n)$ is calculated as follows:

$$\tau_i = m_i(V) + \alpha_V \left(M_i(V) - m_i(V)\right), \tag{10}$$

where

$$\begin{aligned}
M_i(V) &= V(N) - V(N \setminus \{i\}), \\
m_i(V) &= \max_{S \ni i} \left(V(S) - \sum_{j \in S \setminus \{i\}} M_j(V) \right), \\
\alpha_V &= \begin{cases} 0, & M(V) = m(V), \\ \dfrac{\sum_{i \in N} M_i(V) - \sum_{i \in N} m_i(V)}{V(N) - \sum_{i \in N} m_i(V)}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Using (10), we obtain: $\tau = (5\frac{1}{2}, 1, 1\frac{1}{2})$.

In the similar way one can calculate the τ-value $\tau(g^*) = (\tau_1(g^*), \dots, \tau_n(g^*)$ if network g^* is formed at the first stage of the game:

$$\tau_i(g^*) = m_i(v(g^*)) + \alpha_{v(g^*)} \left(M_i(v(g^*)) - m_i(v(g^*))\right), \qquad (11)$$

where

$$\begin{aligned} M_i(v(g^*)) &= v(g^*, N) - v(g^*, N \setminus \{i\}), \\ m_i(v(g^*)) &= \max_{S \ni i} \left(v(g^*, S) - \sum_{j \in S \setminus \{i\}} M_j(v(g^*)) \right), \\ \alpha_{v(g^*)} &= \begin{cases} 0, & M(v(g^*)) = m(v(g^*)), \\ \frac{\sum_{i \in N} M_i(v(g^*)) - \sum_{i \in N} m_i(v(g^*))}{v(g^*, N) - \sum_{i \in N} m_i(v(g^*))}, & \text{otherwise.} \end{cases} \end{aligned}$$

Using (11), we obtain: $\tau(g^*) = (3\frac{1}{2}, 1, 3\frac{1}{2})$.

Note that the τ-value is time-inconsistent since there exists a player $i \in N$ that

$$\tau_i \neq \tau_i(g^*).$$

Nevertheless, one can find time-consistent imputation distribution procedure β for the τ-value (which will also be strongly time-consistent since the τ-value is the single-valued cooperative solution concept):

$$\begin{array}{ll} \beta_{12} = \tau_1(g^*) = 3\frac{1}{2}, & \beta_{11} = \tau_1 - \tau_1(g^*) = 2, \\ \beta_{22} = \tau_2(g^*) = 1, & \beta_{21} = \tau_2 - \tau_2(g^*) = 0, \\ \beta_{32} = \tau_3(g^*) = 3\frac{1}{2}, & \beta_{31} = \tau_3 - \tau_3(g^*) = -2, \end{array}$$

or in the matrix form:

$$\beta = \begin{pmatrix} 2 & 3\frac{1}{2} \\ 0 & 1 \\ -2 & 3\frac{1}{2} \end{pmatrix}.$$

References

- Bala V, Goyal S. A non-cooperative model of network formation. *Econometrica* 2000;**68**(5):1181-1231.
- Dutta B, Van den Nouweland A, Tijs S. Link formation in cooperative situations. *Int J Game Theory* 1998;**27**:245-256.
- Goyal S, Vega-Redondo F. Network formation and social coordination. *Games Econ Behav* 2005;**50**:178-207.
- Jackson M, Watts A. On the formation of interaction networks in social coordination games. *Games Econ Behav* 2002;**41**(2):265-291.
- Jackson M, Wolinsky A. A strategic model of social and economic networks. *J Econ Theory* 1996;**71**:44-74.
- Kuhn HW. Extensive games and the problem of information. In: Kuhn HW, Tucker AW, editors. *Contributions to the theory of games II*. Princeton: Princeton University Press; 1953. p. 193-216.

References

- Petrosyan LA. Stability of solutions in differential games with many participants. *Vestnik Leningradskogo Universiteta. Ser 1. Matematika Mekhanika Astronomiya* 1977;**19**:46-52.
- Petrosjan LA. Cooperative differential games. In: Nowak AS, Szajowski K, editors. *Annals of the International Society of Dynamic Games. Applications to economics, finance, optimization, and stochastic control.* Basel: Birkhäuser; 2005. p. 183-200.
- Petrosyan LA, Danilov NN. Stability of solutions in non-zero sum differential games with transferable payoffs. *Vestnik Leningradskogo Universiteta. Ser 1. Matematika Mekhanika Astronomiya* 1979;**1**:52-59.
- Petrosyan LA, Sedakov AA. Multistage networking games with full information. *Matematicheskaya teoriya igr i ee prilozheniya* 2009;**1**(2):66-81.
- Petrosyan LA, Sedakov AA, Bochkarev AO. Two-stage network games. *Matematicheskaya teoriya igr i ee prilozheniya* 2013;**5**(4):84-104.
- Tijs SH. An axiomatization of the τ-value. *Math Soc Sci* 1987;**13**:177-181.