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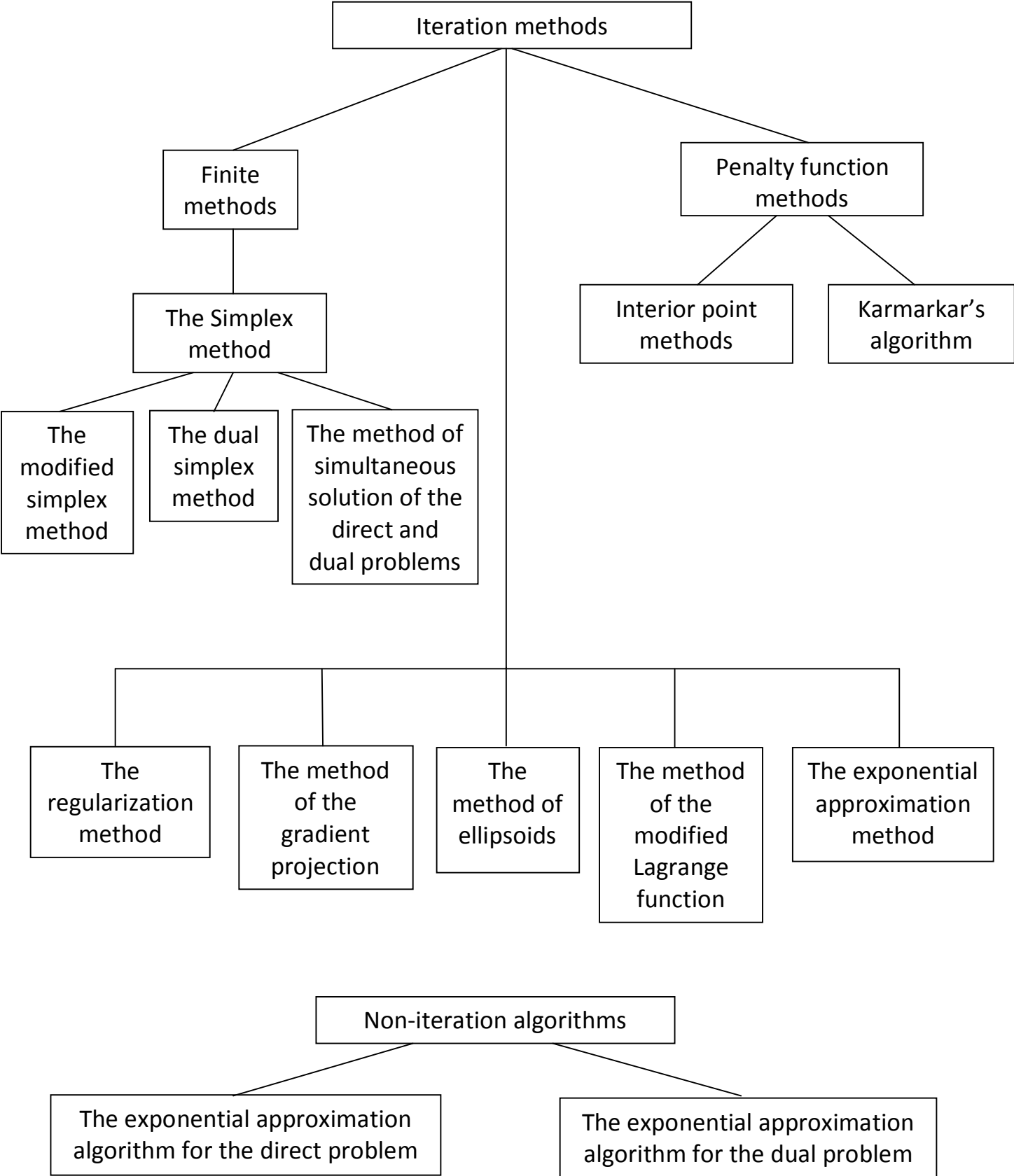
An exponential approximation algorithm in linear programming

Linear programming problems

$$\sum_{j=1}^n c_j x_j \rightarrow \max(\min),$$

$$\sum_{j=1}^n a_{ij} x_j = b_i (\leq b_i, \geq b_i), \quad i = 1, \dots, m, \quad x_j \geq 0, \quad j = 1, \dots, n$$

The classification of methods and algorithms for linear programming



Data preparation and transformation of the original problem

Linear programming problem will be considered as follows:

$$\begin{aligned} \sum_{j=1}^n \hat{c}_j \hat{x}_j &\rightarrow \min, \\ \sum_{j=1}^n \hat{a}_{ij} \hat{x}_j &\geq \hat{b}_i, \quad i=1, \dots, m, \quad \hat{x}_j \geq 0, \quad j=1, \dots, n, \quad n > m, \end{aligned} \quad (1)$$

where $\hat{x}_j \geq 0, j=1, \dots, n$ are the variables of the original problem, $\hat{c}_j, j=1, \dots, n$ are the coefficients of the linear form, $\hat{a}_{ij}, i=1, \dots, m, j=1, \dots, n$ are the coefficients of the constraints and $\hat{b}_i, i=1, \dots, m$ are the right-hand sides of the constraints. It is assumed that (1) has a solution. The corresponding dual problem is

$$\begin{aligned} \sum_{i=1}^m \hat{b}_i \hat{\lambda}_i &\rightarrow \max, \\ \sum_{i=1}^m \hat{a}_{ij} \hat{\lambda}_i &\leq \hat{c}_j, \quad j=1, \dots, n, \quad \hat{\lambda}_i \geq 0, \quad i=1, \dots, m, \end{aligned} \quad (2)$$

where $\hat{\lambda}_i, i=1, \dots, m$ are the values of the original dual problem. Introducing additional non-negative variables $\hat{\lambda}_i \geq 0, i=m+1, \dots, n$, we transform the problem (2) to the canonical form:

$$\begin{aligned} \sum_{i=1}^m \hat{b}_i \hat{\lambda}_i &\rightarrow \max, \\ \sum_{i=1}^m \hat{a}_{ij} \hat{\lambda}_i + \hat{\lambda}_{m+j} &= \hat{c}_j, \quad j=1, \dots, n, \\ \hat{\lambda}_i &\geq 0, \quad i=1, \dots, m+n. \end{aligned} \quad (3)$$

The coefficients of the objective function, the coefficients of the constraints and the right-hand sides of constraints become positive as a result of equivalent transformations:

$$\begin{aligned} \sum_{i=1}^{m+n} \hat{b}_i \hat{\lambda}_i &\rightarrow \max, \\ \sum_{i=1}^{m+n} \hat{a}_{ij} \hat{\lambda}_i &= \hat{c}_j, \quad \hat{c}_j > 0, \quad \hat{a}_{ij} \geq 0, \quad i=1, \dots, m+n, \quad j=1, \dots, n, \\ \hat{\lambda}_i &\geq 0, \quad \hat{b}_i > 0, \quad i=1, \dots, m+n. \end{aligned} \quad (4)$$

Further, the constants and the variables are replaced with some new ones as follows: divide the coefficients, the constraints, and their right-hand sides by \hat{c}_j and

obtain $\sum_{i=1}^{m+n} (\widehat{a}_{ij}/\widehat{c}_i)\widehat{\lambda}_j = 1, j = 1, \dots, n$. Now we replace variables $\widehat{\lambda}_i$ with $\bar{\lambda}_i = \widehat{\lambda}_i/(e\lambda_i^{\max})$, $\lambda_i^{\max} = \max_j(\widehat{c}_j/\widehat{a}_{ij}), (\widehat{a}_{ij} \neq 0), i = 1, \dots, m+n$. After this, the coefficients of the objective function and the constraints are: $\bar{a}_{ij} = (e\widehat{a}_{ij}\lambda_i^{\max})/\widehat{c}_i, \bar{b}_i = e\widehat{b}_i\lambda_i^{\max}, i = 1, \dots, m+n, j = 1, \dots, n$. We replace the coefficients of the objective function with $b_i = \bar{b}_i/b^{\max}, i = 1, \dots, m+n, b^{\max} = \max_i \bar{b}_i$. Next, we replace the variables $\bar{\lambda}_i$ with $\lambda_i = b_i\bar{\lambda}_i$ and obtain: $a_{ij} = \bar{a}_{ij}/b_i, i = 1, \dots, m+n, j = 1, \dots, n$. Finally, the problem (4) can be written in the following form:

$$\begin{aligned} \sum_{i=1}^{m+n} \lambda_i &\rightarrow \min, \\ 0 \leq \lambda_i &\leq e^{-1}, i = 1, \dots, m+n, \\ \sum_{i=1}^{m+n} a_{ij} \lambda_i &= 1, j = 1, \dots, n. \end{aligned} \quad (5)$$

The corresponding dual problem is

$$\begin{aligned} \sum_{j=1}^n x_j &\rightarrow \max, x_j \geq 0, j = 1, \dots, n, \\ y_i = \sum_{j=1}^n a_{ij} x_j - 1 &\leq 0, i = 1, \dots, m+n. \end{aligned} \quad (6)$$

The linear programming problem (5) is replaced with the exponential type approximating function with the positive parameter ε :

$$\begin{aligned} \varphi(\tilde{x}, \varepsilon) &= \sum_{j=1}^n \tilde{x}_j - \varepsilon \sum_{i=1}^{m+n} \exp((\sum_{j=1}^n a_{ij} \tilde{x}_j - 1 - \varepsilon)\varepsilon^{-1}) \rightarrow \max, \\ \tilde{\lambda}_i(\tilde{x}, \varepsilon) &= \exp((\sum_{j=1}^n a_{ij} \tilde{x}_j - 1 - \varepsilon)\varepsilon^{-1}), i = 1, \dots, m+n, \varepsilon > 0. \end{aligned}$$

The exponential transformation of linear programming problems (1) with proofs and examples are described in details in the publications [1-2]. In this paper, the exponential transformation is applied to the dual problem with respect to the problem (1). As $\varepsilon \rightarrow 0$ the optimal value of the approximating function $\varphi(\tilde{x}, \varepsilon)$, the variables $\tilde{\lambda}_i$ and \tilde{x}_j converges to the optimal solutions of the primal (5) and the dual (6) linear problems: $\sum_{i=1}^{m+n} \tilde{\lambda}_i \rightarrow \sum_{i=1}^{m+n} \lambda_i^*, \sum_{j=1}^n \tilde{x}_j \rightarrow \sum_{j=1}^n x_j^*$ and $\tilde{\lambda}_i \rightarrow \lambda_i^*, \tilde{x}_j \rightarrow x_j^*$, where $x_j^*, j = 1, \dots, n, \lambda_i^*, i = 1, \dots, m+n$ are the optimal solutions of the primal (5) and dual (6) linear problems. Since $\varphi(\tilde{x}, \varepsilon)$ is strictly concave, the first order necessary and sufficient conditions for \tilde{x} to be a maximum can be written as follows: $\partial\varphi(\tilde{x}, \varepsilon)/\partial\tilde{x}_j = \sum_{i=1}^{m+n} a_{ij} \tilde{\lambda}_i(\tilde{x}, \varepsilon) - 1 = 0, j = 1, \dots, n$. Here the variables $\tilde{\lambda}_i(\tilde{x}, \varepsilon)$ are the exponential functions of the variables \tilde{x}_j . The system of the necessary and sufficient

conditions can be simplified. Substituting the first order power series expansion of $\tilde{\lambda}_i(\tilde{x}, \varepsilon)$ with respect to $((\sum_{j=1}^n a_{ij}\tilde{x}_j - 1) - \varepsilon)$, $i = 1, \dots, m+n$ for $\tilde{\lambda}_i(\tilde{x}, \varepsilon)$, we get $\sum_{k=1}^n \tilde{x}_k \sum_{i=1}^{m+n} a_{ij} a_{ik} - \sum_{i=1}^{m+n} a_{ij} - \varepsilon = 0$, $j = 1, \dots, n$. As $\varepsilon \rightarrow 0$ we get an approximating system of linear equations to calculate the approximate variables \tilde{x}_j :

$$\sum_{k=1}^n \tilde{x}_k \sum_{i=1}^{m+n} a_{ij} a_{ik} = \sum_{i=1}^{m+n} a_{ij}, \quad j = 1, \dots, n. \quad (7)$$

The variables $\tilde{y}_i = \sum_{j=1}^n a_{ij}\tilde{x}_j - 1 \leq 0$ are the approximation of the constraints $y_i \leq 0$ of the problem (6). If the constraint y_i is active, then \tilde{y}_i reaches its maximum value. Accordingly largest variables \tilde{y}_i define n the optimal basis variables λ_i of the problem (5). On the other hand, smallest variables \tilde{y}_i define m the optimal non-basis variables $\lambda_i = 0$ of the problem (5) and $\hat{\lambda}_i = 0$ of problems (4) and (3). These variables determine the active constraints of the problem (3). The active constraints in turn determine the optimal basis variables of the problem (1).

The description of the algorithm

The algorithm consists the following steps: 1) Convert the original problem (1) to special form (5); 2) Solve the approximating system of linear equations (7) to determine the variables \tilde{x}_j ; 3) Calculate the approximating values of constraints \tilde{y}_i ; 4) Identify the active constraints and the optimal basis variables of the original problem (1); 5) Calculate the values of the optimal basis variables of the problem (1).

A numerical example

Application of the algorithm is illustrated by the following numerical example:

$$\begin{aligned} 8\hat{x}_1 + 2\hat{x}_2 + 2\hat{x}_3 + \hat{x}_4 &\rightarrow \min, \\ \hat{x}_1 + 3\hat{x}_2 + 0,2\hat{x}_3 + \hat{x}_4 &\geq 2, \\ 4\hat{x}_1 - 2\hat{x}_2 + \hat{x}_3 - 0,5\hat{x}_4 &\geq 4, \\ \hat{x}_j &\geq 0, \quad j = 1, \dots, 4. \end{aligned} \quad (8)$$

The solution of the original linear problem is $\sum_{j=1}^4 \hat{c}_j \hat{x}_j^* = 9,7143$, $(\hat{x}_j^*) = (1,1429 \ 0,2857 \ 0,0 \ 0,0)$. The corresponding dual problem is

$$\begin{aligned} 2\hat{\lambda}_1 + 4\hat{\lambda}_2 &\rightarrow \max, \\ \hat{\lambda}_1 + 4\hat{\lambda}_2 &\leq 8, \\ 3\hat{\lambda}_1 - 2\hat{\lambda}_2 &\leq 2, \\ 0,2\hat{\lambda}_1 + \hat{\lambda}_2 &\leq 2, \\ \hat{\lambda}_1 - 0,5\hat{\lambda}_2 &\leq 1, \\ \hat{\lambda}_i &\geq 0, \quad i=1,2. \end{aligned} \tag{9}$$

Now we introduce the additional non-negative variables $\hat{\lambda}_i \geq 0, i=3, \dots, 6$ to transform the problem to the canonical form:

$$\begin{aligned} 2\hat{\lambda}_1 + 4\hat{\lambda}_2 &\rightarrow \max, \\ \hat{\lambda}_1 + 4\hat{\lambda}_2 + \hat{\lambda}_3 &= 8, \\ 3\hat{\lambda}_1 - 2\hat{\lambda}_2 + \hat{\lambda}_4 &= 2, \\ 0,2\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_5 &= 2, \\ \hat{\lambda}_1 - 0,5\hat{\lambda}_2 + \hat{\lambda}_6 &= 1, \\ \hat{\lambda}_i &\geq 0, \quad i=1,2. \end{aligned} \tag{10}$$

Next, using equivalent transformations, we transform the problem to the form:

$$\begin{aligned} 8,2\hat{\lambda}_1 + 18,5\hat{\lambda}_2 + 6\hat{\lambda}_3 + \hat{\lambda}_4 + \hat{\lambda}_5 + \hat{\lambda}_6 &\rightarrow \min, \\ \hat{\lambda}_1 + 4\hat{\lambda}_2 + \hat{\lambda}_3 &= 8, \\ 4\hat{\lambda}_1 + 2\hat{\lambda}_2 + \hat{\lambda}_3 + \hat{\lambda}_4 &= 10, \\ 0,2\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_5 &= 2, \\ 2\hat{\lambda}_1 + 3,5\hat{\lambda}_2 + \hat{\lambda}_3 + \hat{\lambda}_6 &= 9, \\ \hat{\lambda}_i &\geq 0, \quad i=1, \dots, 6. \end{aligned} \tag{11}$$

Finally, replacing of constants and variables, we transform the problem to the form:

$$\begin{aligned} \sum_{i=1}^6 \lambda_i &\rightarrow \min, \quad 0 \leq \lambda_i \leq e^{-1}, \quad i=1, \dots, 6, \\ 3,8329\lambda_1 + 6,7957\lambda_2 + 5,2384\lambda_3 &= 1, \\ 12,2654\lambda_1 + 2,7183\lambda_2 + 4,1907\lambda_3 + 25,1441\lambda_4 &= 1, \\ 3,0664\lambda_1 + 6,7957\lambda_2 + 125,7205\lambda_5 &= 1, \\ 6,8141\lambda_1 + 5,2855\lambda_2 + 4,6563\lambda_3 + 27,9379\lambda_6 &= 1. \end{aligned} \tag{12}$$

The approximating system (7) has the form:

$$\begin{aligned} 88,3134\tilde{x}_1 + 87,4376\tilde{x}_2 + 57,9348\tilde{x}_3 + 86,4286\tilde{x}_4 &= 15,867, \\ 87,4376\tilde{x}_1 + 807,6175\tilde{x}_2 + 56,0828\tilde{x}_3 + 117,4588\tilde{x}_4 &= 44,3185, \\ 57,9348\tilde{x}_1 + 56,0828\tilde{x}_2 + 15861,2369\tilde{x}_3 + 56,8135\tilde{x}_4 &= 135,5826, \\ 86,4286\tilde{x}_1 + 117,4588\tilde{x}_2 + 56,8135\tilde{x}_3 + 876,5766\tilde{x}_4 &= 44,6939. \end{aligned}$$

The solution is $(\hat{x}_j) = (0,1022 \ 0,0381 \ 0,0079 \ 0,0353)$. The approximating values of the constraints are $(\hat{y}_i) = (0,1241 \ 0,0384 \ -0,1406 \ -0,0412 \ -0,0051 \ -0,0141)$. We will put these values in the descending order: $\hat{y}_1 = 0,1241$, $\hat{y}_2 = 0,0384$, $\hat{y}_5 = -0,0051$, $\hat{y}_6 = -0,0141$, $\hat{y}_4 = -0,0412$, $\hat{y}_3 = -0,1406$. The optimal basis variables of the problems (10), (11), (12) correspond to largest $\tilde{y}_1, \tilde{y}_2, \tilde{y}_5, \tilde{y}_6$. The optimal non-basis variables correspond to smallest \hat{y}_3, \hat{y}_4 . These variables are additional for the first two inequalities of problem (10). The values of $\hat{y}_3 = 0$, $\hat{y}_4 = 0$ indicate that the first two inequalities are the active constraints of the problem (9). These inequalities transformed to a system of two equations with two variables $\hat{\lambda}_1, \hat{\lambda}_2$. The variables \hat{x}_1, \hat{x}_2 of the problem (8) correspond to the two equations of the problem (9). These variables need to be determined by solving the system of two equations with two variables. This is possible if the variables \hat{x}_3, \hat{x}_4 are the optimal non-basis variables, that are $\hat{x}_3 = 0$, $\hat{x}_4 = 0$. Finally, we find the optimal values of the basis variables of the problem (8) by solving the system:

$$\begin{aligned} \hat{x}_1 + 3\hat{x}_2 &= 2, \\ 4\hat{x}_1 - 2\hat{x}_2 &= 4. \end{aligned}$$

The solution is $\sum_{j=1}^4 \hat{c}_j \hat{x}_j^* = 9,7143$, $(\hat{x}_j^*) = (1,1429 \ 0,2857 \ 0,0 \ 0,0)$. Thus, we find the variables of the optimal basis solution by solving the approximating system of the linear equations with four constraints and four variables. To calculate the optimal values of the basic variables we solve the system of the linear equations with two variables.

Conclusion

To compare, we solve the problem (8) by the simplex method, which is an iterative algorithm. The introduction of additional variables and the transition to maximization leads to the canonical form:

$$\begin{aligned} -8\hat{x}_1 - 2\hat{x}_2 - 2\hat{x}_3 - \hat{x}_4 &\rightarrow \max, \\ \hat{x}_1 + 3\hat{x}_2 + 0,2\hat{x}_3 + \hat{x}_4 - \hat{x}_5 &= 2, \\ 4\hat{x}_1 - 2\hat{x}_2 + \hat{x}_3 - 0,5\hat{x}_4 - \hat{x}_6 &= 4, \\ \hat{x}_j &\geq 0, \quad j=1, \dots, 4. \end{aligned}$$

The maximum number of iterations for solving the simplex method is calculated by formula $c_6^2 = 6! / (2!(6-2)!) = 15$. This number depends essentially on the size of the problem. The actual number of iterations is a priori unknown. Let us take variables \hat{x}_3, \hat{x}_4 as the starting basis. We transform the problem so that the coefficients of these variables would form the unit matrix.

Table 1. The first iteration

\hat{x}_j basis	\hat{c}_j basis	\hat{b}_i	-8 \hat{x}_1	-2 \hat{x}_2	-2 \hat{x}_3	-1 \hat{x}_4	0 \hat{x}_5	0 \hat{x}_6
\hat{x}_3	-2	4,5455	4,0909	-0,4546	1	0	-0,4546	-0,9091
\hat{x}_4	-1	1,0909	0,1818	3,0909	0	1	-0,9091	0,1818
Criterion K			-0,3636	-0,1818	0	0	1,8182	1,6364

The initial basis solution is $(\hat{x}_j) = (0, 0, 4,4555, 1,0909, 0, 0, 0, 0)$. The first value of the objective function is $\sum_{j=1}^4 \hat{c}_j \hat{x}_j = -10,1818$. The minimum negative evaluation is $K = -0,3636$, therefore the variable x_1 is introduced into the basis. The first column is the key column. The minimum ratio \hat{b}_i / \hat{a}_{i1} , $\hat{a}_{i1} > 0$ corresponds to the first row. Hence the first row is the key row. The variable \hat{x}_3 is derived from the basis variables. Now we divide the elements of the key row for the key element $\hat{a}_{11} = 4,0909$ and add to the second row elements of the transformed key row, multiplied by $(-1)\hat{a}_{21} = 0,1818$. New basis solution is obtained.

Table 2. The second iteration

\hat{x}_j basis	\hat{c}_j basis	\hat{b}_i	-8 \hat{x}_1	-2 \hat{x}_2	-2 \hat{x}_3	-1 \hat{x}_4	0 \hat{x}_5	0 \hat{x}_6
\hat{x}_1	-8	1,1111	1	-0,1111	0,2444	0	-0,1111	-0,2222
\hat{x}_4	-1	0,8889	0	3,1111	-0,0444	1	-0,8889	0,2222
Criterion K			0	-0,2222	0,0889	0	1,7778	1,5556

The solution is $\sum_{j=1}^4 \hat{c}_j \hat{x}_j = -9,7778$, $(\hat{x}_j) = (1,1111, 0, 0, 0, 0, 8889, 0, 0, 0, 0)$.

Table 3. The third iteration

\hat{x}_j	\hat{c}_j	\hat{b}_i	-8	-2	-2	-1	0	0
basis	basis		\hat{x}_1	\hat{x}_2	\hat{x}_3	\hat{x}_4	\hat{x}_5	\hat{x}_6
\hat{x}_1	-8	1,1429	1	0	0,2429	0,0357	-0,1429	-0,2143
\hat{x}_2	-2	0,2857	0	1	-0,0143	0,3214	-0,2857	0,0714
Criterion K			0	0	0,0857	0,0714	1,7143	1,5714

All the values K are non-negative here. The optimal solution is $\sum_{j=1}^4 \hat{c}_j \hat{x}_j^* = -9,7143$, $(\hat{x}_j^*) = (1,1429 \ 0,2857 \ 0,0 \ 0,0 \ 0,0 \ 0,0)$. The advantage of the non-iteration algorithm is that the amount of computation is defined a priori by the dimension of the problem.

Sources

1. Gordunovsky V.M. *Exponential approximation method for linear programming*. Moscow: Editus; 2013 (in russian).
2. Gordunovsky V.M. Exponential approximation algorithm for solving linear programming problems. *Proceedings of the VII Moscow International Conference on Operations Research*. Vol. 1. Moscow: MAKS Press; 2013. p. 61-64.

Thank you